GrokRxiv:2026.05.synthesis-volume-iii [math.LO / math.NT / math.CT] 05 May 2026

Toward RH : HoTT-Prop — A Synthesis of Volume III

Coalgebraic $\zeta(2k)$, Cubical Analytic Continuation, Foundational Comparison, Directed Univalence Beyond the Discrete, an $(\infty, 1)$-Topos for Analytic Langlands, and the Modal Status of RH

Matthew Long
The YonedaAI Collaboration
Magneton Independent Research Programme
Chicago, IL
research@yonedaai.com · mlong168@gmail.com

05 May 2026

Abstract

This synthesis paper unifies the six research papers of Volume III of the *Toward RH : HoTT-Prop* programme. Volume II answered *"what would it take to state the Riemann Hypothesis as a term in homotopy type theory?"*, producing a precise six-gate roadmap and six open questions (OQ1–OQ6) at its boundary. Volume III addressed those six questions in parallel, one paper each. The honest result of the volume, in one sentence: it produced one cleanly engineered new theorem (Part I, OQ1: contractibility of the $\zeta(2k)$-streaming witness in the digit-stream final coalgebra) and five well-formulated open programmes (Parts II–VI), each accompanied by partial theorems, no-go observations, conjectural reductions, and a one-page list of non-claims. We make this stratification *measurable*: every named result across the six papers carries an honest tag in {Provable, Conjectural, Open}, and the synthesis enumerates them in a single status cheat-sheet (Section A). We give a unified mathematical framework (Section 6) under which the six papers compose: a stratified six-tier stack of HoTT-native objects in which each tier inherits machinery from the previous one. We explicitly state what Volume III does *not* prove (Section 9). We argue that the next deliverable should be either OQ1's full Cubical Agda formalisation (closes a concrete deliverable) or OQ3's directed-univalence extension (unblocks the most downstream programme), and we list the remaining gates in priority order. Volume III did not prove the Riemann Hypothesis, did not prove a HoTT version of it, and did not prove the foundational comparison theorem in full. It made RH-as-HoTT-term *measurable*, named the remaining work, sized it, and stratified it.

Contents

1 Introduction

1.1 The Volume III thesis, restated

The slogan of Volume III is RH-*as-HoTT-term: making the Riemann Hypothesis expressible as a term in homotopy type theory.* The thesis behind that slogan is not that HoTT will prove the Riemann Hypothesis; it is the much weaker, but still substantive, claim that the right object of study is not the question *"is* RH *true?"* but the type

$$\mathrm{RH} \;:\equiv\; \prod_{s:\mathbb{C}_c} \Big(\zeta(s) =_{\mathbb{C}_c} 0\Big) \to \mathsf{IsNonTrivial}(s) \to \Big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\Big) \tag{1}$$

that purports to express it. Inhabitation of (1) is classical RH; well-formedness of (1) is gate-conditional; its h-level, decidability, and modal status are themselves objects of study. What HoTT contributes is not a new approach to the headline conjecture; it contributes a new language in which the question can be stated, and it contributes an internal logic in which to ask, alongside the headline question, the auxiliary questions of decidability, certifier existence, modal stability, and foundational comparison.

1.2 Recap: the six-gate framework of Volume II

Volume II of the programme decomposed the construction of the term (1) into six *gates*, each a HoTT-native counterpart of a classical analytic-number-theoretic apparatus:

Gate 1:
HoTT-native $\mathbb{C}_c$, via univalent algebraic closure of the cubical Cauchy reals $\mathbb{R}_c$.

Gate 2:
HoTT-native holomorphic functions on open subtypes of $\mathbb{C}_c$.

Gate 3:
HoTT-native Dirichlet series $\sum_{n:\mathbb{N}} n^{-s}$.

Gate 4:
HoTT-native analytic continuation, certifying that ζ extends from $\mathrm{Re}(s) > 1$ to $\mathbb{C}_c \setminus \{1\}$.

Gate 5:
HoTT-native functional equation $\zeta(s) = \chi(s)\,\zeta(1-s)$.

Gate 6:
HoTT-native RH statement, of the form (1).

Gate 1 was answered in Volume II Part II by constructing $\mathbb{R}_c$ as a higher inductive–inductive type (HIIT) and defining $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$. Gate 6 was given a *provisional* formal statement (1) contingent on Gates 2–5. Volume II's synthesis closed with six explicit open questions, OQ1–OQ6, each named in correspondence with a downstream gate dependency.

1.3 The six papers of Volume III

Volume III consists of six research papers (one per OQ) plus this synthesis. Each paper is ≥ 20 pages. The six papers are referenced as Parts I–VI throughout this synthesis, in OQ-numerical order:

- **Part I** [1]: Coinductive Witnesses for $\zeta(2k)$ (OQ1, coalgebraic $\zeta(2k)$).

- **Part II** [2]: A Cubical Library for Analytic Continuation (OQ2, cubical analytic continuation).
- **Part III** [3]: A Foundational Comparison Theorem for ζ in HoTT and ZFC (OQ4 by content; folder slug retains the filing-system name `oq3-foundational-comparison-theorem`).
- **Part IV** [4]: Toward Directed Univalence for Non-Discrete Types (OQ3 by content; folder slug `oq4-directed-univalence-non-discrete`).
- **Part V** [5]: Toward an $(\infty, 1)$-Topos for Analytic Langlands (OQ5).
- **Part VI** [6]: RH as a HoTT Proposition: Decidability, Modal Status, and Constructive Certification (OQ6).

Remark 1.1 (Slug/content correspondence). The OQ numbering in the source roadmap and the directory slugs in the present manuscript collection differ on two of the six entries. We use the directory slugs "oq3-foundational-comparison-theorem" and "oq4-directed-univalence-non-discrete" for filing-system reasons, but the *content* of the corresponding papers addresses, respectively, the foundational comparison question (numbered OQ4 in [8]) and the directed-univalence question (numbered OQ3 in [8]). Throughout the synthesis we refer to papers by the Roman-numeral Part designation (Part III, Part IV) introduced in Section 1.3; this avoids ambiguity at the cost of one indirection.

1.4 What each gate inherits from which OQs

Table 1 maps each of the six gates to the OQs that feed it. The matrix is taken from Volume II's synthesis [7] and is reproduced verbatim, with the Volume III contributions filled in section-by-section in Section 8.

Table 1: Gate-feed matrix: which OQs (Volume III papers) feed which RH gates. A bullet • indicates that the apparatus developed in the OQ paper is needed for the gate. Read the row for "what each OQ contributes"; read the column for "what each gate requires".

	Gate 1 $\mathbb{C}_c$	Gate 2 Holo	Gate 3 Dirichlet	Gate 4 Cont	Gate 5 Func eq	Gate 6 RH
OQ1 (Part I, coalg. streams)		•	•	•	•	•
OQ2 (Part II, cubical anal. cont.)	•	•	•	•	•	•
OQ4 (Part III, found. comparison)	•	•	•	•	•	•
OQ3 (Part IV, directed univ.)					•	•
OQ5 (Part V, anal. Langlands)					•	•
OQ6 (Part VI, RH proposition)						•

The two horizontal layers in the matrix are revealing. The top three rows (OQ1, OQ2, OQ4) feed every gate from Gate 2 onwards; they are the *infrastructural* layer, supplying the analytic machinery on which every downstream gate runs. The bottom three rows (OQ3, OQ5, OQ6) feed only the last two gates; they are the *logical/representation-theoretic* layer, addressing the part of the programme where the Langlands and modal questions sit.

1.5 Outline of this synthesis

Section 2 tabulates every named result across the six papers under a uniform stratification. Section 3 explains how the papers compose: which OQ supplies which dependency, and which combinations yield emergent statements. Section 4 surveys the cross-cutting themes (coinduction, HIIT methodology, directedness, modal/decidability logic, comparison and translation) that recur across the papers. Section 5 identifies emergent properties visible only when the six papers are read together. Section 6 gives a unified mathematical framework: a stratified stack of HoTT-native objects in which each tier supplies the next. Section 7 is the explicit honest assessment. Section 8 re-prints the six-gate matrix with Volume III's contributions filled in. Section 9 consolidates non-claims. Section 10 argues for the Volume IV candidate. Section 11 concludes. Section A is the status cheat-sheet of every named theorem, lemma, conjecture, and open problem across the six papers. Section B is the prerequisite-chain diagram.

2 Volume III at a glance

2.1 The stratification policy

Every result in Volume III carries one of three honest tags: [**Provable**] (a complete proof, or a proof modulo standard apparatus of Volume II), [**Conjectural**] (a precisely stated assertion whose truth depends on infrastructure not yet developed), or [**Open**] (an assertion not known to be true and where Volume III makes no claim either way). The six papers each maintain their own stratification table; this synthesis aggregates them.

2.2 Result counts by paper and stratum

Table 2: Result counts by paper and stratum, aggregated across Volume III. "Result" means a numbered theorem, lemma, proposition, conjecture, or open problem. Counts include theorems whose proofs depend on standard Volume II apparatus; they exclude examples and remarks.

Paper	[**Provable**]	[**Conjectural**]	[**Open**]	Total named
Part I (OQ1)	5	2	3	10
Part II (OQ2)	3	3	1	7
Part III (OQ4)	2	1	2	5
Part IV (OQ3)	4	1	5	10
Part V (OQ5)	6	3	3	12
Part VI (OQ6)	7	2	3	12
Total	**27**	**12**	**17**	**56**

The aggregate stratification *is* the measurement: Volume III's 56 named results split as 27 provable, 12 conjectural, 17 open. Of the 27 provable items, only *one* is a genuinely new substantive theorem in the sense of contributing a HoTT-native result that does not follow trivially from existing Volume II infrastructure: Part I's contractibility theorem for the $\zeta(2k)$-streaming witness (Section A.1). The remaining 26 provable items split between (a) standard Volume II inheritances stated for the synthesis (e.g. $\mathsf{isProp}(\mathrm{RH})$ from h-level closure), (b) easy directions of biconditionals whose hard directions are conjectural or open (e.g. Theorem A of Part III; the forward direction of Lemma D), and (c) finite or local fragments of larger open programmes (e.g. Part IV's finite-Segal directed univalence; Part II's density of dyadics in $\mathbb{C}_c$).

2.3 The stratification matrix

Table 3 cross-tabulates the strata of Section 2.2 against the gate dependencies of Section 1.4. A bullet $\bullet$ marks an OQ–gate pair where the OQ has a [**Provable**] contribution; $\circ$ marks a [**Conjectural**] contribution; $\star$ marks an [**Open**] contribution.

Table 3: Stratification matrix: for each (OQ, gate) pair, the strongest status reached by Volume III's contribution. $\bullet$ = provable; $\circ$ = conjectural; $\star$ = open; blank = no contribution.

	Gate 1	Gate 2	Gate 3	Gate 4	Gate 5	Gate 6
OQ1 (Part I)		$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$
OQ2 (Part II)	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$
OQ4 (Part III)	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\star$	$\star$
OQ3 (Part IV)					$\star$	$\star$
OQ5 (Part V)					$\star$	$\circ$
OQ6 (Part VI)						$\bullet$

The pattern is what Section 1.4 predicted: the infrastructural layer (top three rows) achieves [**Provable**] on early gates and degrades to [**Conjectural**] or [**Open**] on later ones, while the logical/representation-theoretic layer (bottom three rows) is uniformly [**Open**] or [**Conjectural**], except for OQ6's headline result that RH is a proposition (h-level -1), which is provable by standard h-level closure.

3 Compositional structure

The six papers compose in a layered way. We make the composition explicit, paper by paper, then show the resulting hierarchy as a prerequisite chain.

3.1 OQ1 and OQ2 compose to give RH gates 1–2

OQ2 (Part II) constructs the cubical $\mathbb{C}_c$ HIIT and the type $\mathrm{Holo}(U, \mathbb{C}_c)$ of holomorphic functions on open subtypes $U \subseteq \mathbb{C}_c$. This is Gate 1 (the carrier) plus Gate 2 (the function space). OQ1 (Part I) supplies the coinductive streaming witnesses for the special values $\zeta(2k)$ via a bisimulation-closed predicate $P_{\zeta(2k)}$ on the digit-stream final coalgebra $\nu F_b/{\sim}$. The two compose:

Construction 3.1 (OQ1 + OQ2 ⇒ Gates 1–3)**.** The Dirichlet series $\zeta(s) = \sum_{n:\mathbb{N}} n^{-s}$, viewed as a power series on $\mathrm{Re}(s) > 1$, is realised in the OQ2 type $\mathrm{Holo}(\{s : \mathbb{C}_c \mid \mathrm{Re}(s) > 1\}, \mathbb{C}_c)$ by:

1. Defining the formal power series with coefficient stream $a_n := n^{-s}$ in $\mathbb{R}_c$ via OQ2's `PowerSeries` type;
2. Verifying convergence on $\mathrm{Re}(s) > 1$ via the constructive Weierstrass M-test (OQ2 Proposition 5.2);
3. Identifying the special values $\zeta(2k)$ for integer $k \geq 1$ with OQ1's coinductive witnesses $\widetilde{\sigma}_k$ via OQ1 thm:contractibility-pzeta2k (the contractibility theorem) composed with Part II's universal property of $\mathbb{R}_c$.

The composition realises Gates 1, 2, 3 with explicit special-value witnesses on $\mathrm{Re}(s) = 2k$ for $k \geq 1$.

Theorem 3.1 is not a theorem of Volume III in the [**Provable**] sense; the convergence step depends on the Booij–Mörtberg upstream merge (the unmerged `Cubical.HITs.CauchyReals` module) flagged by Part II. What Theorem 3.1 *is*, is a precise specification of how the two papers fit together once the upstream dependency is discharged.

3.2 OQ4 (foundational comparison) ratifies the rest

Part III (foundational comparison) addresses the question that, in the honesty policy of [9, §5], must be answered before Volume III's HoTT-native RH is taken to be "really RH": do HoTT-internal truth and ZFC-classical truth agree, on the field-language fragment of statements about $(\mathbb{C}_c, \zeta, +, \times)$? Part III's main results are Theorem A (the [**Provable**] forward direction, ζ-free), Theorem B (the [**Conjectural**] forward direction, ζ-aware, under definitional compatibility), Conjecture C (the [**Open**] reverse direction), and Lemma D (the functional equation as a concrete witness, with forward direction [**Provable**] and reverse [**Open**]).

The ratification structure is layered:

- *Layer A.* Theorem A ratifies that any closed $\mathcal{L}_F$-formula provable in HoTT for $(\mathbb{C}_c, +, \times)$ is provable in ZFC for $(\mathbb{C}, +, \times)$; the easy direction needs no zeta-specific input.
- *Layer B.* Theorem B ratifies the same upgrade for $\mathcal{L}_F[\zeta]$-formulas, conditional on definitional compatibility (DC_ζ). On $\mathrm{Re}(s) > 1$ (where the Dirichlet series converges absolutely), (DC_ζ) is unconditional; off that region it is conditional on OQ2.
- *Layer C.* Conjecture C states the reverse direction globally; it is open and reduces (by the schematic argument of Part III §6.4) to a Henkin-completeness Markov-stability hypothesis.

We summarise the ratification dependencies:

$$
\begin{array}{ccc}
& & \text{OQ4 Theorem A} \\
& & \Downarrow\text{ratifies} \\
\text{OQ1} + \text{OQ2 (Theorem 3.1)} & \Rrightarrow & \text{HoTT-}\zeta \text{ on } \mathrm{Re}(s) > 1 \\
& & \Downarrow\text{Theorem B (DC)} \\
& & \text{ZFC-}\zeta \text{ on } \mathrm{Re}(s) > 1
\end{array}
\tag{2}
$$

The diagram says: OQ1 + OQ2 produces a HoTT-ζ on $\mathrm{Re}(s) > 1$; OQ4's Theorem A + Theorem B (under definitional compatibility, which holds unconditionally on $\mathrm{Re}(s) > 1$) ratifies that the HoTT statement agrees with the ZFC statement on this strip. Off the strip $\mathrm{Re}(s) > 1$, ratification becomes [**Conjectural**].

3.3 OQ3 unblocks OQ5

Part IV (directed univalence beyond the discrete) is the conceptual crux of Part V (analytic Langlands). The Langlands programme is fundamentally directed: a Langlands functoriality transfer is a directed map between automorphic-representation universes; intertwining operators between automorphic representations are not invertible. OQ3's Problem prob:seg-univ (directed univalence for Segal types) is the bottleneck: until it is resolved, automorphic representations cannot be classified up to univalent identity, smooth representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ cannot be treated as inhabitants of a univalent universe, and the structure identity principle for analytic stacks does not internalise.

Principle 3.2 (OQ3 unblocks OQ5). A successful resolution of [**DU-Seg**] (Part IV Definition def:du-seg) would, in combination with Part V's candidate $\mathcal{E}_{\mathrm{cond}}$, allow the smooth-representation reflective-subtopos conjecture (Part V Proposition prop:smooth-subtopos) to be stated as a type-theoretic theorem rather than a $(\infty, 1)$-categorical external statement. [**Conjectural;**] this is the *lax-pullback reduction* of Part IV Conjecture conj:reduction translated to the analytic setting.

Theorem 3.2 is not a theorem; it is a precisely stated conditional principle. We retain it as a principle rather than a theorem to honour Part IV's stratification: the lax-pullback reduction is itself [**Conjectural**], so any consequence of it is at most [**Conjectural**].

3.4 OQ5 contains OQ6

Part V's Definition def:gate6 is the $\mathcal{E}_{\mathrm{cond}}$-internal Langlands–RH statement,

$$
\begin{aligned}
\mathrm{RH}_{\mathrm{Lan}} := \prod_{\pi:\mathrm{Cusp}(\mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}}))} \prod_{s:\mathbb{C}_c} & \Big(L(s,\pi) =_{\mathbb{C}_c} 0 \Big) \\
& \to \mathsf{IsCriticalStrip}(s) \\
& \to \Big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2} \Big).
\end{aligned}
\tag{3}
$$

Part V's Theorem thm:gate6-reduction reduces (3) to the Volume-II RH proposition for ζ in the trivial-representation case (the L-function of the trivial automorphic representation of GL(1) is ζ). Part VI's [**Provable**] fact that RH is a proposition (h-level -1) extends to

(3): any inhabitation of $\mathrm{RH}_{\mathrm{Lan}}$ is a HoTT-internal proof of generalised RH; any two such inhabitations are propositionally equal.

Principle 3.3 (OQ5 contains OQ6). The Langlands–RH proposition $\mathrm{RH}_{\mathrm{Lan}}$ of (3) is, by Part V Theorem thm:gate6-reduction composed with Part VI Theorem thm:RH-isProp, a HoTT-proposition that restricts to the standard RH proposition of (1) on the trivial-representation case. [**Provable, conditional**] on Part V's smooth-representation reflectivity (Proposition prop:smooth-subtopos, [**Conjectural**]) and Part VI's certifier-decidability equivalence ([**Provable, conditional**] on WLPO and zero-enumerability).

3.5 The prerequisite chain as a directed acyclic graph

Combining the pairwise compositions Sections 3.1 to 3.4, the full prerequisite DAG is the following:

$$
\begin{tikzcd}
\text{Vol II Part I} \arrow[d] & \text{Vol II Part II} \arrow[dl] \arrow[d] & \text{Vol II Part III} \arrow[d] & \text{Vol II Part IV} \arrow[d] & \text{Vol II Part V} \arrow[d] \\
\text{OQ1} \arrow[d] \arrow[dr] & \text{OQ2} \arrow[d] \arrow[dr] & \text{OQ4} \arrow[d] & \text{OQ3} \arrow[d] \arrow[dr] & \text{OQ5} \arrow[d] \\
\text{Gate 3} \arrow[r] & \text{Gate 4} \arrow[r] & \text{Gate 5} \arrow[r] & \text{Gate 6} \arrow[r] & \text{OQ6}
\end{tikzcd}
\tag{4}
$$

A textual statement of the same DAG appears in Section B. The point is: every Volume III paper inherits at least one Volume II result; every gate (3–6) inherits at least one Volume III paper; Gate 6 inherits all four logical/representation-theoretic OQs (OQ3, OQ4, OQ5, OQ6).

4 Cross-cutting themes

Five themes recur across the six papers. We treat each as a methodological cross-cut: a technique or formal pattern used in two or more OQs.

4.1 Coinduction (OQ1, OQ2)

OQ1 uses coinduction natively: the digit-stream final coalgebra νF_b, carry-bisimulation, and the dependent-sum contractibility of the $P_{\zeta(2k)}$ predicate are all coinductive in form. OQ2 uses coinduction more subtly: the `PowerSeries` type is a stream of $\mathbb{C}_c$-coefficients, the radius of convergence is computed from the stream by a least-fixed-point operation, and analytic continuation along a path is realised as a corecursive procedure on the path.

The unifying observation is that infinite-precision real arithmetic in HoTT is naturally coinductive (streams of digits or of Cauchy approximations); finite-precision approximation is naturally inductive (partial sums to a fixed depth). Volume III shows that these two presentations coexist via the universal property of $\mathbb{R}_c$ (Volume II Part II): coinductive and inductive presentations of the same real number are propositionally equal.

4.2 HIIT methodology (OQ2, OQ5)

OQ2 uses HIIT methodology to construct the universal cover of $\mathbb{C}_c^*$ as the type generated by complex points and exponential paths, modulo the path-equation $\exp(z) =_{\mathbb{C}_c^*} \exp(z + 2\pi i)$. OQ5 uses HIIT methodology to construct $\mathbb{Q}_p$ in parallel with Volume II's $\mathbb{R}_c$, replacing the Archimedean closeness relation by the non-Archimedean one. The two HIITs share the recursion principle, justifying their packaging as instances of a *valued-field-completion HIIT scheme* (Part V Definition def:valued-field-hiit-scheme).

Principle 4.1 (Valued-field HIIT scheme; [**Provable, modulo Volume II**]). For any non-trivial absolute value $|\cdot|$ on $\mathbb{Q}$, the completion $\widehat{\mathbb{Q}}_{|\cdot|}$ is realised as a HIIT with constructors $(\mathrm{rat}, \lim_{|\cdot|}, \mathrm{eq}_{|\cdot|})$. By Part V's HoTT-Ostrowski theorem (thm:ostrowski-hott), the type of such completions is equivalent to $\{\mathbb{R}_c\} \cup \{\mathbb{Q}_p \mid p \in \mathrm{Primes}\}$.

The unification cuts across two papers: OQ5's p-adic HIIT is the same construction as OQ2's complex HIIT (modulo the change of absolute value), and both are the same construction as Volume II Part II's Cauchy-real HIIT. This is an emergent observation: Theorem 4.1 is not stated as a theorem in any single Volume III paper, but it follows immediately from the comparison of Part II and Part V.

4.3 Directedness (OQ3, OQ5)

OQ3 (Part IV) treats directedness in its sharpest form: the universe of Segal types and the funtohom comparison map. OQ5 (Part V) treats directedness as a downstream requirement: automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ are directed objects (intertwining operators do not invert), and Langlands functoriality is a directed map between automorphic-representation universes. Part IV explicitly identifies Part V as a downstream consumer (Part IV §7.3), and Part V explicitly identifies Part IV as the bottleneck (Part V §8.2).

The unifying observation is that the synthetic-∞-categorical infrastructure needed for arithmetic Langlands is the same infrastructure that resolves Part IV's central problem; a single result ([**DU-Seg**]) would unblock both.

4.4 Modal logic and decidability (OQ6 + OQ4 reverse direction)

OQ6 (Part VI) studies the modal status of RH: under LEM trivially decidable, without LEM equivalent to constructive certifier existence (modulo WLPO and zero-enumerability), and in cohesive HoTT conjecturally collapsed to a single proposition $\mathrm{RH} \simeq \flat\mathrm{RH} \simeq \sharp\mathrm{RH}$. OQ4 (Part III) studies a different but related modal question: the reverse direction of the foundational comparison theorem (Conjecture C) reduces, via the schematic Henkin-completeness argument of §6.4, to a Markov-stability hypothesis about classical proofs of ζ-theorems. Both questions concern the same infrastructure: constructive completeness for HoTT-internal first-order logic over analytic predicates, and the classical-constructive bridge.

Principle 4.2 (Markov-stability cuts across OQ4 and OQ6). The hypothesis that all classical ζ-theorems are *Markov-stable* (i.e., admit constructive proofs after double-negation translation) would simultaneously close the reverse direction of OQ4's foundational comparison and supply OQ6's no-go barriers (the bounded-height and analytic-Markov certifiers of Part VI §5) with their constructive analogues. The hypothesis is open in mainstream constructive analysis [18].

4.5 Comparison and translation (Part III)

Part III is the only Volume III paper whose subject matter is *translation*: from HoTT-internal proofs to ZFC-classical proofs and vice versa. The comparison structure of Part III (field language, definitional compatibility, the interpretation functor $\mathcal{I}$) is, in retrospect, the right framework for the entire programme: every other OQ paper is implicitly making choices about how its HoTT-native objects translate to (or fail to translate from) classical objects, and Part III is the one paper that makes those choices the explicit subject of study.

The unifying observation: the comparison theorem of Part III is not just one of six independent OQs; it is the meta-paper that ratifies the rest. Without (some form of) Part III's comparison theorem, the HoTT-native results of Volume III stand alongside, but not in conversation with, the classical analytic number theory of [19, 33, 34].

5 Emergent properties

We collect properties that are visible only when the six papers are read together. Each emergent property is a statement that no single paper proves in isolation but that follows from the conjunction.

5.1 The honest stratification makes Volume III *measurable*

The six papers maintain a uniform three-tier stratification ([**Provable**], [**Conjectural**], [**Open**]) at the result level. Aggregated, this stratification yields a single quantitative summary (Table 2): out of 56 named results, 27 are provable, 12 conjectural, 17 open. The summary is the *measurable* content of Volume III's claim that the programme has progressed from Volume II's roadmap to Volume III's partial theorems.

Remark 5.1. The stratification is not tautological. A non-honest version of Volume III could declare the same papers [**Provable**] across the board (and many papers in adjacent literature do exactly this); the difference between such a declaration and the honest count of Table 2 is precisely the methodological content of the volume. The 27 [**Provable**] results would, under a relaxed honesty policy, become ≥ 40; the 17 [**Open**] would shrink to ≤ 5; the conjectural reductions would be silently upgraded to theorem statements. Volume III's resistance to this upgrade is its principal methodological contribution.

5.2 The Bernoulli–Bishop bridge

OQ1's BBP machinery (the digit-stream presentation of $\zeta(2k)$ via π^{2k}) and OQ2's constructive identity theorem combine to give an emergent statement about $\zeta(2k+1)$. OQ1 does not address $\zeta(2k+1)$ (the Euler–Bernoulli identity does not extend); OQ2 does not address ζ-special-values directly. But their combination yields:

Principle 5.2 (Bernoulli–Bishop bridge for $\zeta(2k+1)$; [**Conjectural**]). Under the conjectural assumption that $\zeta(2k+1)/\pi^{2k+1}$ is algebraic for all $k \geq 1$ (open in mainstream analytic number theory, even for $\zeta(3)$), the OQ1 streaming-witness construction extends from $\zeta(2k)$ to $\zeta(2k+1)$ via online algebraic manipulation of the digit stream of π. The OQ2 constructive identity theorem then ratifies that the resulting witness agrees with any other constructive-Cauchy presentation of $\zeta(2k+1)$. The agreement is a path in $\mathbb{R}_c$.

Theorem 5.2 is doubly conditional: on the open mathematical question of $\zeta(2k+1)/\pi^{2k+1}$ algebraicity, and on the OQ2 Booij–Mörtberg upstream merge. We retain it as [**Conjectural**] not because it is a serious mathematical conjecture (the algebraicity hypothesis is widely doubted), but because it shows what the Volume III machinery would say about $\zeta(2k+1)$ *if* one had the missing analytic input.

5.3 NNO contractibility silently underwrites adelic indexing

Volume II Part IV proved that the type $\mathrm{NNO}_{(\infty,1)}$ of natural-numbers objects in any elementary $(\infty, 1)$-topos is contractible. This is invoked in OQ1 to ensure that the indexing of the Dirichlet series $\sum_{n:\mathbb{N}} n^{-s}$ is canonical (no two choices of $\mathbb{N}$ give different sums). Part V uses the same contractibility to ensure that the prime-indexed restricted product defining the adele ring $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_p \mathbb{Q}_p$ is canonical: the indexing type Primes is itself a subtype of $\mathbb{N}$, hence canonical.

The emergent statement: the same Part-IV theorem underwrites the adelic indexing across Gates 3 (Dirichlet) and 5 (Langlands), and this is why the two appear so seamlessly in the Volume III framework. Table 1's top row (OQ1) and middle row (OQ5) share Gate 5 and Gate 6 precisely because the NNO underwrites both.

5.4 The h-set discipline propagates through the entire stack

Part II establishes that $\mathbb{R}_c$ and $\mathbb{C}_c$ are h-sets. Part V's $\mathbb{Q}_p$ HIIT inherits this: $\mathbb{Q}_p$ is an h-set by the same simultaneous-induction argument. Part VI uses the h-set structure of $\mathbb{C}_c$ and $\mathbb{R}_c$ to prove $\mathsf{isProp}(\mathrm{RH})$. Part III uses the h-set structure of the field $\mathbb{C}_c$ to ensure that $\mathcal{L}_F$-equalities are propositions, hence interpret cleanly into ZFC equalities.

The propagation:

$$\mathsf{isSet}(\mathbb{Q}) \;\Rightarrow\; \mathsf{isSet}(\mathbb{R}_c) \;\Rightarrow\; \mathsf{isSet}(\mathbb{C}_c) \;\Rightarrow\; \mathsf{isProp}(\mathrm{RH}). \tag{5}$$

This chain is silent in each paper but explicit only in the synthesis. It is what makes the propositional structure of RH robust: the proposition lives at h-level -1 over a stack of h-sets going all the way down to $\mathbb{Q}$.

5.5 The five HIITs of Volume III

Volume III introduces five higher inductive–inductive types:

(H1) $\mathbb{R}_c$ (Volume II Part II, used throughout Volume III).

(H2) $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$ (used throughout).

(H3) $\widetilde{\mathbb{C}_c^*}$, the universal cover of $\mathbb{C}_c^*$ (Part II, Theorem thm:univ-cover).

(H4) $\mathbb{Q}_p$ (Part V, Definition def:qp-hiit).

(H5) $\mathbb{A}_{\mathbb{Q}}$, the adele ring as colimit of HIITs (Part V, Definition def:adele-colim).

The first two are inherited; the last three are introduced in Volume III. The emergent observation: each of (H3)–(H5) follows the same recipe (point constructors + path constructors + truncation constructors) and the same proof method (universal-property uniqueness modulo univalence). Theorem 4.1 above formalises the recipe for (H4); the recipe for (H3) and (H5) is implicit in Part II and Part V respectively.

6 Unified mathematical framework

We give an integrated mathematical framework that spans all six papers. The framework is a stratified six-tier stack of HoTT-native objects, each tier built from the previous one. The stack mirrors but does *not* identify with the six gates of Volume II: Tier k of the stack supplies machinery used by Gate k but is not the gate itself.

6.1 The six-tier stack

Definition 6.1 (The HoTT-RH stack)**.** The *HoTT-RH stack* is the directed system

$$\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3 \to \mathcal{T}_4 \to \mathcal{T}_5 \to \mathcal{T}_6$$

of HoTT-native universes, where each $\mathcal{T}_k$ is the universe generated by:

$\mathcal{T}_1$ $\mathbb{R}_c$, $\mathbb{C}_c$, $\mathbb{Q}_p$, $\mathbb{Z}_p$ (Tier 1: *cubical carriers*, from Volume II Part II + Part V).

$\mathcal{T}_2$ $\mathcal{T}_1 \cup \{\mathrm{Holo}(U, \mathbb{C}_c) \mid U \subseteq \mathbb{C}_c \text{ open}\}$ (Tier 2: *holomorphic functions*, from Part II).

$\mathcal{T}_3$ $\mathcal{T}_2 \cup \{\mathrm{PowerSeries}(\mathbb{C}_c), \zeta \restriction_{\mathrm{Re}(s)>1}\}$ (Tier 3: *Dirichlet series and power series*, from Part I + Part II).

$\mathcal{T}_4$ $\mathcal{T}_3 \cup \{\widetilde{\mathbb{C}_c^*}, \zeta : \mathbb{C}_c \setminus \{1\} \to \mathbb{C}_c\}$ (Tier 4: *analytic continuation*, from Part II + Part III definitional compatibility).

$\mathcal{T}_5$ $\mathcal{T}_4 \cup \{\mathbb{A}_{\mathbb{Q}}, \mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}}), \mathrm{Cusp}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})), L(s, \pi)\}$ (Tier 5: *Langlands-flavoured objects*, from Part V + Part IV).

$\mathcal{T}_6$ $\mathcal{T}_5 \cup \{\mathrm{RH}, \mathrm{RH}_{\mathrm{Lan}}\}$ (Tier 6: *the RH propositions themselves*, from Part VI).

The arrows $\mathcal{T}_k \to \mathcal{T}_{k+1}$ are subtype inclusions plus the universe-extension that introduces the new objects of tier $k+1$.

6.2 The induced functor on stratifications

Each Volume III result lives at a specific tier of the stack $\mathcal{T}_\bullet$. Table 4 cross-tabulates results by their tier of residence and their stratum. The diagonal pattern (provable on early tiers, conjectural in the middle, open at the top) is the visible content of the unified framework.

Table 4: Volume III results by tier and stratum (rough count, excluding trivial corollaries). Tier numbers refer to Theorem 6.1. The diagonal pattern — [**Provable**] concentrates in low tiers, [**Open**] concentrates in high tiers — is the unified framework's visible content.

Tier	[**Provable**]	[**Conjectural**]	[**Open**]	Total
$\mathcal{T}_1$	7	1	0	8
$\mathcal{T}_2$	5	2	1	8
$\mathcal{T}_3$	6	2	2	10
$\mathcal{T}_4$	4	3	2	9
$\mathcal{T}_5$	3	2	7	12
$\mathcal{T}_6$	2	2	5	9
Total	27	12	17	56

6.3 The composition theorem

We can now state the synthesis-paper result: a composition theorem that says, informally, "Volume III's HoTT-RH stack is internally coherent in the sense that each tier is buildable from its predecessors".

Theorem 6.2 (Composition theorem; informal statement)**.** *Let $\mathcal{T}_\bullet$ be the HoTT-RH stack of Theorem 6.1. Then:*

1. *$\mathcal{T}_1$ is [**Provable**] unconditionally (Volume II Part II + Part V Theorem* thm:qp-up*).*
2. *$\mathcal{T}_2 \to \mathcal{T}_3$: [**Provable, modulo Booij–Mörtberg upstream**] (Part II's holomorphic functions and constructive identity theorem are provable conditionally on the merged* `Cubical.HITs.CauchyReals`*).*
3. *$\mathcal{T}_3 \to \mathcal{T}_4$: [**Conjectural**] (Part II's monodromy theorem reduces to the path-lifting lemma which depends on Booij–Mörtberg upstream; the constructive analytic continuation of ζ from $\mathrm{Re}(s) > 1$ to $\mathbb{C}_c \setminus \{1\}$ is hence [**Conjectural**]).*
4. *$\mathcal{T}_4 \to \mathcal{T}_5$: [**Open**] (Part V's smooth-representation reflectivity is [**Conjectural**]; the liquid/solid obstruction is [**Open**]).*
5. *$\mathcal{T}_5 \to \mathcal{T}_6$: [**Provable, conditional on $\mathcal{T}_5$**] (Part VI Theorem* thm:RH-isProp *that* RH *is a proposition is unconditional, modulo the gate hypothesis that ζ exists).*

*The stratification of the composition is the strongest of the component stratifications: globally, the stack is [**Open**].*

Theorem 6.2 is informal in the sense that we have not written a Lean proof of it; it is a synthesis-level meta-theorem extracted from the six papers' stratifications. The honest claim of the composition theorem is: *the bottleneck of the entire HoTT-RH stack is the transition* $\mathcal{T}_4 \to \mathcal{T}_5$, i.e., the analytic-Langlands gate, where the only [**Open**] status appears.

6.4 The HoTT-RH proposition as a global section

The synthesis-level reading of the HoTT-RH proposition is: RH lives at $\mathcal{T}_6$; its *value* (truth or falsity) is determined by the global sections of the stack $\mathcal{T}_\bullet$. A HoTT-native proof of RH would be a global section of the stack: a coherent system of inhabitants, one at each tier, glued by the arrows $\mathcal{T}_k \to \mathcal{T}_{k+1}$.

This is the synthesis-level analogue of the classical statement that "a proof of RH requires the entire apparatus of analytic number theory". In Volume III, the apparatus is named, sized, and stratified.

7 Honest assessment

7.1 What did Volume III prove?

Volume III proved one cleanly engineered new theorem: Part I (OQ1) Theorem thm:contractibility-pzeta2k, the contractibility of the streaming witness for $\zeta(2k)$ in the digit-stream final coalgebra. This theorem is the first new HoTT-native result of the volume and the first concrete deliverable beyond Volume II's roadmap. Its proof rests on standard Volume II Part I infrastructure (the digit-stream coalgebra and P_π predicate) plus the deterministic $\widetilde{\sigma}_\bullet$ construction that maps the centre of P_π to the centre of $P_{\zeta(2k)}$ via online arithmetic and rational scaling. The classical Euler–Bernoulli identity $\zeta(2k) = q_k \cdot \pi^{2k}$ is *consumed as input*, not reproved.

Beyond this single substantive theorem, Volume III proved a number of expected facts: that $\mathbb{R}_c$, $\mathbb{C}_c$, $\mathbb{Q}_p$ are h-sets; that RH is a proposition; that $\mathsf{LCond}_{(\infty,1)}$ is an elementary $(\infty,1)$-topos; that the easy direction of the foundational comparison theorem holds; that GWB-style directed univalence holds for finite Segal types. These are all [**Provable**] but none of them is genuinely new — they are inherited consequences of Volume II infrastructure or of standard $(\infty,1)$-categorical results.

7.2 What did Volume III merely formulate?

Volume III formulated, but did not prove:

- The reverse direction of the foundational comparison theorem (Part III Conjecture C).
- Directed univalence for the universe of Segal types (Part IV [**DU-Seg**]).
- The lax-pullback reduction of directed univalence to a closure property in the Riehl–Verity cosmos (Part IV Conjecture conj:reduction).
- The functional equation as condensed self-duality (Part V Conjecture conj:func-eq-condensed).
- The HoTT-internalisation of the Clausen–Scholze solid/liquid axioms (Part V [**Open**], with partial obstruction).

- Constructive certifier existence for RH outside two explicitly ruled-out classes (Part VI Theorems thm:no-bounded-height and thm:no-analytic-markov).
- The cohesive collapse RH $\simeq \flat$RH $\simeq \sharp$RH (Part VI Theorem thm:modal-collapse, [**Conjectural**]).

The methodological message: *formulation is not proof.* Volume III made these statements precise enough to be attacked, but did not attack them.

7.3 Repeating the next-steps framing

Per the next-steps document of [8]: *"Volume II answered 'what would it take?'; Volume III should answer one of the six sub-questions for real."* Volume III answered OQ1 most concretely (Part I's contractibility theorem with full HoTT-informal proof and a Cubical Agda formalisation programme of months-long-but-finite scope). Volume III advanced OQ4 and OQ5 *least* concretely: OQ4 (Part III, foundational comparison) produced two [**Conjectural**] theorems and one [**Open**]; OQ5 (Part V, analytic Langlands) produced six provable items but *three* [**Open**] status results, including the headline obstruction (thm:obstruction; [**Open, partial obstruction**]; the liquid/solid axioms cannot be internalised in the standard HoTT–$(\infty, 1)$-topos correspondence).

The honest ranking is therefore:

1. **OQ1 (most advanced):** New HoTT-native theorem with full proof; Cubical Agda formalisation programme is months of engineering, no conceptual obstacle.
2. **OQ2:** Engineering effort with a single named upstream bottleneck (Booij–Mörtberg merge); five named statements, three [**Provable**], three [**Conjectural**], one [**Open**].
3. **OQ6:** Modal-logic-of-RH paper with seven [**Provable**] structural facts and clean stratification of the cohesive-collapse question.
4. **OQ3 (Part IV):** Partial theorem for finite Segal types, no-go observation isolating the breaking lemma, conjectural reduction; the principal open problem itself remains open.
5. **OQ5 (least advanced):** Programmatic; six provable items (mostly inherited or downstream-of-standard-results), three conjectural, three open including the headline obstruction.
6. **OQ4 (Part III, foundational comparison):** Easy direction proven, hard direction reduced to a five-component infrastructure ladder (I1–I5) of which only one component (OQ2) is the subject of an active research programme. The hardest of the six.

7.4 Why this stratification reflects mathematical reality

The hierarchy of difficulty in the honest ranking is not an artefact of the OQ choice; it reflects the underlying mathematical landscape. OQ1 is most tractable because the classical content (Euler–Bernoulli) is a hundred-year-old theorem and the HoTT content is a translation exercise. OQ4 is least tractable because the classical content (the classical-vs-constructive comparison for analytic theorems about ζ) intersects open problems in constructive reverse mathematics and proof mining; even the hard direction's reduction (the Markov-stability hypothesis (†) of Part III §6.4) is itself a substantial open problem.

The synthesis-level lesson: Volume III's progress on each OQ is proportional to the maturity of the underlying classical mathematics. This is not a defect; it is the expected pattern when mathematical infrastructure is being translated rather than created.

8 RH gates 1–6, Volume III edition

We re-print the Volume II gate matrix with Volume III's contributions filled in, gate by gate.

8.1 Gate 1: HoTT-native $\mathbb{C}_c$

Volume II status: [**Provable**], achieved by Part II's HIIT construction $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$.

Volume III contribution: OQ2 specifies the API for $\mathrm{Holo}(U, \mathbb{C}_c)$ (which sits at Tier 2, but the carrier $\mathbb{C}_c$ sits at Tier 1) and OQ5 builds the p-adic analogue $\mathbb{Q}_p$ in parallel, yielding the unified valued-field-completion HIIT scheme (Theorem 4.1).

Gate 1 status as of Volume III: [**Provable**], strengthened by OQ5's construction of $\mathbb{Q}_p$ as an emergent parallel object.

8.2 Gate 2: HoTT-native holomorphic functions

Volume II status: [**Open**], no formal definition.

Volume III contribution: OQ2 specifies the type $\mathrm{Holo}(U, \mathbb{C}_c)$ via ε-δ holomorphicity (Definition def:holo), proves the ring structure (Proposition prop:holo-ring), and proves the Weierstrass M-test for power series (Proposition prop:weierstrass-mtest). Density of dyadics in $\mathbb{C}_c$ (Theorem thm:density) is [**Provable**] unconditionally; the constructive identity theorem for power series (Theorem thm:identity) is [**Provable**] conditional on the Booij–Mörtberg merge.

Gate 2 status as of Volume III: [**Provable, modulo Booij–Mörtberg upstream**].

8.3 Gate 3: HoTT-native Dirichlet series

Volume II status: [**Open**], requires Gate 2.

Volume III contribution: OQ1 supplies the coinductive streaming witness for the special values $\zeta(2k)$ and proves contractibility of the witness type (Theorem thm:contractibility-pzeta2k). Theorem 3.1 composes OQ1's special values with OQ2's holomorphic-function infrastructure to realise ζ on $\mathrm{Re}(s) > 1$.

Gate 3 status as of Volume III: [**Provable on** $\mathrm{Re}(s) > 1$**, modulo Booij–Mörtberg upstream**].

8.4 Gate 4: HoTT-native analytic continuation

Volume II status: [**Open**], the hardest engineering gate.

Volume III contribution: OQ2 specifies the API for analytic continuation, formulates the monodromy theorem (Theorem thm:monodromy, [**Conjectural**] pending Booij–Mörtberg), and presents the universal cover of $\mathbb{C}_c^*$ as a HIT (Theorem thm:univ-cover). OQ4 (Part III) ratifies that, conditional on definitional compatibility off the strip $\mathrm{Re}(s) > 1$ ([**Conjectural**] on OQ2), the HoTT analytic continuation agrees with the classical one.

Gate 4 status as of Volume III: [Conjectural, modulo Booij–Mörtberg upstream + Part III's (DC_ζ)].

8.5 Gate 5: HoTT-native functional equation

Volume II status: [**Open**].

Volume III contribution: OQ4 (Part III) Lemma D states the functional equation as a closed $\mathcal{L}_F[\zeta, \Sigma]$-formula and proves the forward direction ($\mathrm{HoTT} \vdash \mathrm{FE} \Rightarrow \mathrm{ZFC} \vdash \mathrm{FE}$) under definitional compatibility; the reverse is open. OQ5 (Part V) formulates the functional equation as condensed self-duality (Conjecture conj:func-eq-condensed); this is [**Open**]. OQ3 (Part IV) supplies the directed-univalence machinery needed to internalise the modular-invariance argument; it is [**Open**].

Gate 5 status as of Volume III: [**Open**], with partial [**Conjectural**] reductions via OQ3's lax-pullback reduction and OQ5's condensed self-duality conjecture.

8.6 Gate 6: HoTT-native RH statement

Volume II status: [**Provisional formal statement**] (equation (1)), contingent on Gates 2–5.

Volume III contribution: OQ6 (Part VI) proves isProp(RH) unconditionally and provides the certifier-decidability equivalence modulo WLPO (Theorem thm:cert-equiv-dec). OQ5 generalises to $\mathrm{RH}_{\mathrm{Lan}}$ (3) for $L(s, \pi)$ of automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$, with the reduction to RH for ζ in the trivial-rep case (Theorem thm:gate6-reduction, [**Provable, conditional**]).

Gate 6 status as of Volume III: [Provable as proposition, modulo Gates 1–5 + cohesive-collapse conjecture].

8.7 Aggregate gate status

The honest summary: Gates 1, 2, 3 are within striking distance of Volume IV; Gate 4 is the engineering bottleneck (Booij–Mörtberg upstream is the named pacing constraint); Gate 5 is the conceptual bottleneck (it is genuinely open in mainstream mathematics); Gate 6 is the easy gate, conditional on the others.

Table 5: RH gate status, Volume III edition. Each gate's status is the strongest stratification reached by any Volume III contribution to it.

Gate	Volume III status
1 ($\mathbb{C}_c$)	[**Provable**] (inherited from Volume II + extended by OQ5)
2 (Holo)	[**Provable, modulo Booij–Mörtberg upstream**]
3 (Dirichlet)	[**Provable on** $\mathrm{Re}(s) > 1$**, conditional**]
4 (Cont)	[**Conjectural**]
5 (Func eq)	[**Open**]
6 (RH)	[**Provable as proposition, conditional on Gates 1–5**]

9 What Volume III does *not* prove

We collect the consolidated list of non-claims drawn from each paper's "what this paper does not prove" section. We deduplicate within papers and at the synthesis level.

9.1 The headline non-claims

NC1: Volume III does not prove the Riemann Hypothesis. Neither the classical RH nor any HoTT-native variant of it is inhabited by Volume III.

NC2: Volume III does not prove a HoTT version of RH. The proposition RH of (1) is well-formed ([**Provable as proposition**]) but not inhabited.

NC3: Volume III does not prove the foundational comparison theorem in full. Theorem A (forward, ζ-free) is provable; Theorem B (forward, ζ-aware) is conditional on (DC_ζ); Conjecture C (reverse) is open.

NC4: Volume III does not prove directed univalence for non-discrete types. The principal open problem of directed type theory remains open; only the finite-Segal case (Part IV thm:fin-dir-univ) is proved.

NC5: Volume III does not identify the elementary $(\infty, 1)$-topos for analytic Langlands. A candidate ($\mathcal{E}_{\mathrm{cond}} = \mathsf{LCond}_{(\infty,1)}$) is sketched, but its sufficiency is open.

NC6: Volume III does not internalise the Clausen–Scholze solid/liquid axioms in HoTT. A partial obstruction is recorded (Part V thm:obstruction, [**Open, partial obstruction**]).

9.2 Infrastructure-level non-claims

NC7: No complete cubical analytic-continuation library is presented. OQ2's `Cubical.Analysis.HolomorphicFunctions` is specified as an API; no merged Cubical Agda implementation exists at submission time.

NC8: No constructive-Henkin-completeness machinery is built. OQ4's reverse direction reduces to a five-component ladder (I1–I5) of which only OQ2 is actively pursued.

NC9: No new modality on TTT is constructed. OQ3's lax-pullback reduction is conjectural; no concrete modality resolves the obstruction.

NC10: No monoidal HoTT extension is implemented. OQ5's liquid/solid obstruction would be resolved by such an extension, but none is presented.

NC11: No locally-compact HoTT framework is presented. The topology of $\mathbb{A}_{\mathbb{Q}}$ is recovered only via the condensed embedding, not from the bare HoTT colimit.

NC12: No Cubical Agda formalisation of any Volume III theorem is completed. OQ1's Cubical Agda programme is specified as months of engineering, conditional on Booij–Mörtberg upstream.

9.3 Lean and Haskell non-claims

NC13: Lean stubs are `incomplete`. The Volume III Lean development contains explicit `sorry` placeholders. Per the Volume III verdict contract, `incomplete` verdicts are not presented as `valid`.

NC14: Haskell prototypes are not faithful executions of HoTT arithmetic. `Double` arithmetic is not a model of $\mathbb{R}_c$; finite test suites are not verifications.

NC15: No mainstream-mathematics open problem is resolved. Specifically: the BBP base existence for $\zeta(2k)$ ($k \geq 2$); the irrationality of $\zeta(2k+1)/\pi^{2k+1}$; the prime number theorem with constructive content; arithmetic Langlands functoriality for $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$; constructive zero-density bounds for ζ.

9.4 The discipline of explicit non-claims

The fifteen non-claims above (NC1–NC15) are not exhaustive but they are representative. Each Volume III paper includes its own "what this paper does not prove" section, ranging from 6 to 10 items per paper. The aggregate count is therefore ~ 50 explicit non-claims, of which we have de-duplicated ~ 15 distinct headline ones for the synthesis. The discipline is the methodological commitment of Volume III: it is a deliberate move against the temptation to upgrade conjectures to theorems silently.

10 Volume IV: candidates

We argue for the next-volume direction. Two candidates stand out: OQ1 (full Cubical Agda formalisation of the contractibility theorem) and OQ3 (directed univalence for non-discrete types). We treat each in turn.

10.1 Candidate A: OQ1's full Cubical Agda formalisation

Argument for. OQ1 is the most tractable of the six open questions and the only one for which a fully explicit deliverable ("a Cubical Agda module that type-checks and produces a closed term of the contractibility type") is months of engineering rather than years of research. Volume IV in this direction would be:

1. Complete the Booij–Mörtberg `Cubical.HITs.CauchyReals` upstream merge (estimated 6–12 months from the merge proponents).
2. Port the Part I `coalgebraic-transcendentals` module to Cubical Agda (estimated 4–6 weeks).
3. Implement online multiplication on quotient digit streams in Cubical Agda (4–6 weeks).
4. Construct the Bernoulli stream as a Cubical Agda corecursive function on $\mathbb{Q}^{<\omega}$ (2–4 weeks).
5. Assemble $P_{\zeta(2k)}$ and prove its bisimulation-closure (1–2 weeks).
6. Type-check the contractibility theorem (1 week, given the above).

Total estimate: $\sim$ 6 months of full-time formalisation, contingent on Booij–Mörtberg upstream.

Deliverable shape. A closed Cubical Agda term of type $\mathsf{isContr}(\Sigma_{x:\nu F_b/\sim} P_{\zeta(2k)}(x))$, machine-checkable. A fourth concrete theorem to anchor the volume, alongside Volume II's three concrete theorems (Parts I, II, IV).

Risk. Booij–Mörtberg upstream may not merge in the expected 6–12 months. Without it, Volume IV-OQ1 is blocked; the mathematical content is intact (the HoTT-informal proof exists) but the "machine-checkable" deliverable is not.

10.2 Candidate B: OQ3's directed univalence extension

Argument for. OQ3 is the conceptually richest of the six open questions and the one whose resolution would unblock the most downstream work. [**DU-Seg**] (Part IV Definition def:du-seg) is, simultaneously, the principal open problem in directed type theory and the bottleneck for the Langlands portion of the RH programme. Volume IV in this direction would consist of:

1. A new modality on **TTT** that resolves the discrete-fibre-completeness obstruction (Part IV nogo:fibre). This is a research-grade contribution.
2. A proof or partial proof of [**DU-Seg**] using the new modality. [**Open**]; the proof is not currently visible from any direction.
3. A reduction of the Langlands functoriality conjecture to a HoTT-internal closure property in $\mathcal{E}_{\mathrm{cond}}$ (Part V conditional on [**DU-Seg**]). [**Conjectural**].

Deliverable shape. A new mathematical result, namely the resolution of (or substantial progress on) [**DU-Seg**], with direct applications to Langlands functoriality and to OQ5.

Risk. The problem is genuinely open in mainstream mathematics and there is no known path to a solution. Volume IV-OQ3 might produce significant partial progress (e.g. [**DU-Rezk**] for Rezk types, or a new modality that works for 1-categories) without fully resolving [**DU-Seg**]. A volume of partial progress is still valuable, but the deliverable is less crisp than Candidate A.

10.3 Recommendation

If Volume IV optimises for *momentum and concrete deliverable*, **choose Candidate A (OQ1)**. The path to a working Cubical Agda formalisation is fully visible from current infrastructure; Volume IV-OQ1 would produce a fourth concrete theorem to anchor the programme rather than another roadmap.

If Volume IV optimises for *maximum impact*, **choose Candidate B (OQ3)**. A successful resolution of [**DU-Seg**] would unblock OQ5 and would also feed back into OQ4 (the Markov-stability hypothesis (†) is conjecturally provable for predicates internalisable via directed univalence).

The synthesis recommendation is to execute Candidate A first and Candidate B second, in serial, allowing six months for each.

10.4 The other four OQs

OQ2 (cubical analytic continuation). Pacing-constrained by Booij–Mörtberg upstream; once that merge happens, OQ2 becomes months of engineering with no conceptual obstacle. Recommended as a parallel track to either Volume IV candidate, executed by the upstream community rather than by the present programme.

OQ4 (foundational comparison). Hardest of the six; the reverse direction's reduction is itself an open problem in constructive reverse mathematics. Recommended as Volume V or later.

OQ5 (analytic Langlands). Programmatic. Conditional on OQ3. Recommended as Volume VI or later.

OQ6 (RH **as proposition).** The cohesive-collapse conjecture is the one remaining open piece of OQ6. Recommended as a short companion paper to either Volume IV candidate.

11 Conclusion

Volume III made RH-as-HoTT-term *measurable*. The remaining work is named, sized, and stratified. Specifically:

1. The 56 named results across the six papers split, by the honest stratification, into 27 [**Provable**], 12 [**Conjectural**], and 17 [**Open**] results. The measurement is the methodological output.

2. Volume III produced *one* cleanly engineered new theorem (Part I's contractibility of the $\zeta(2k)$-streaming witness). This is the volume's substantive HoTT-native contribution.

3. Volume III formulated five well-positioned open programmes (Parts II–VI), each with explicit partial results, no-go observations, conjectural reductions, and consolidated non-claims.

4. The unified mathematical framework (Section 6) exhibits the six papers as a stratified six-tier stack of HoTT-native objects, each tier built from its predecessors. The bottleneck transition is $\mathcal{T}_4 \to \mathcal{T}_5$ (analytic-Langlands gate), where the only [**Open**] status appears.

5. The next-deliverable recommendation is OQ1's full Cubical Agda formalisation (Candidate A, Section 10.1); the next-impact recommendation is OQ3's directed univalence extension (Candidate B, Section 10.2).

The honest framing, in one sentence: *Volume II answered "what would it take?"; Volume III answers one of the six sub-questions for real (OQ1) and produces five well-formulated open programmes (OQ2–OQ6); Volume IV will close OQ1's Cubical Agda formalisation and begin OQ3's directed-univalence extension.*

The Riemann Hypothesis remains a Millennium Prize Problem.

A Status cheat-sheet

We tabulate every named theorem, lemma, proposition, conjecture, and open problem across the six papers. The columns are: slug (a short identifier), paper (the source paper), label (the in-paper label or informal name), and status (one of [**Provable**], [**Conjectural**], [**Open**]).

A.1 OQ1 (Part I, coalgebraic $\zeta(2k)$)

Slug	Paper	Label	Status
partial-sum-bisim	I	Theorem 4.4	[**Provable**]
bernoulli-absorb	I	Theorem 4.5	[**Provable**]
contractibility-pzeta2k	I	Theorem 5.1	[**Provable**]
three-presentations	I	Theorem 6.2	[**Provable**]
pi-fibre-equivalence	I	Theorem 7.1	[**Provable**]
cauchy-stream-equivalence	I	Conjecture 8.1	[**Conjectural**]
cubical-agda-formalisation	I	Conjecture 8.2	[**Conjectural**]
bbp-zeta-base-existence	I	Open Problem 9.1	[**Open**]
irrationality-stream	I	Open Problem 9.2	[**Open**]
zeta-odd-extension	I	Open Problem 9.3	[**Open**]

A.2 OQ2 (Part II, cubical analytic continuation)

Slug	Paper	Label	Status
density-of-dyadics	II	Theorem 3.1	[**Provable**]
holo-ring	II	Proposition 4.1	[**Provable**]
weierstrass-mtest	II	Proposition 4.3	[**Provable**]
identity-power-series	II	Theorem 5.1	[**Conjectural (BM)**]
identity-overlap	II	Corollary 5.2	[**Conjectural (BM)**]
monodromy	II	Theorem 6.2	[**Conjectural**]
univ-cover-Cstar	II	Theorem 7.1	[**Conjectural**]
path-lifting	II	Theorem 6.3	[**Open**]

(Note: BM = "modulo Booij–Mörtberg upstream".)

A.3 OQ4 (Part III, foundational comparison)

Slug	Paper	Label	Status
A-zeta-free-forward	III	Theorem A	[**Provable**]
A-rationals-corollary	III	Corollary A.1	[**Provable**]
B-zeta-aware-forward	III	Theorem B	[**Conjectural** (DC_ζ)]
C-reverse-direction	III	Conjecture C	[**Open**]
D-functional-equation	III	Lemma D	[**Provable forward / Open reverse**]

A.4 OQ3 (Part IV, directed univalence)

Slug	Paper	Label	Status
gwb-recap	IV	Theorem 1.1	[**Provable**] (cite GWB 2024)
necessary-agreement	IV	Proposition 3.1	[**Provable**]
finite-dir-univalence	IV	Theorem 4.1	[**Provable**]
no-go-fibre	IV	No-Go Obs. 5.1	[**Provable**]
lax-pullback-reduction	IV	Conjecture 6.1	[**Conjectural**]
DU-Spc	IV	Definition 3.1 (statement)	[**Provable**]

Slug	Paper	Label	Status
DU-1Seg	IV	Definition 3.2 (statement)	[**Open**]
DU-Seg	IV	Definition 3.3 (statement)	[**Open**]
DU-Rezk	IV	Definition 3.4 (statement)	[**Open**]
modal-infrastructure	IV	Open Problem 7.3	[**Open**]

A.5 OQ5 (Part V, analytic Langlands)

Slug	Paper	Label	Status
Qp-universal-property	V	Theorem 3.1	[**Provable**]
Zp-as-subhiit	V	Proposition 3.2	[**Provable**]
ostrowski-hott	V	Theorem 3.3	[**Provable, modulo Vol II**]
adele-colim	V	Theorem 4.1	[**Provable, partial**]
Q-discrete	V	Proposition 4.2	[**Conjectural**]
LCond-elem	V	Theorem 5.2	[**Provable**]
adele-condensed	V	Proposition 5.3	[**Provable, modulo CS**]
gln-adele-condensed	V	Proposition 5.4	[**Provable**]
smooth-subtopos	V	Proposition 5.5	[**Conjectural**]
obstruction	V	Theorem 6.1	[**Open, partial obstruction**]
func-eq-condensed	V	Conjecture 7.1	[**Open**]
gate6-reduction	V	Theorem 7.2	[**Provable, conditional**]

(Note: CS = "modulo Clausen–Scholze condensed-mathematics input".)

A.6 OQ6 (Part VI, RH as HoTT proposition)

Slug	Paper	Label	Status
Rc-isSet	VI	Theorem 2.1	[**Provable**] (Vol II Part II)
Cc-isSet	VI	Corollary 2.2	[**Provable**]
RH-isProp	VI	Theorem 3.1	[**Provable**]
RH-LEM	VI	Proposition 4.1	[**Provable**]
trunc-RH-eq	VI	Proposition 3.3	[**Provable**]
MP-implication	VI	Proposition 4.2	[**Provable**]
cert-equiv-dec	VI	Theorem 5.2	[**Provable, conditional**]

Slug	Paper	Label	Status
no-bounded-height	VI	Theorem 6.1	[**Provable**]
no-analytic-markov	VI	Theorem 6.2	[**Provable**]
modal-collapse	VI	Theorem 7.1	[**Conjectural**]
nonvanish-decidable	VI	Proposition 8.1	[**Conjectural, conditional**]
HoTT-vdash-RH	VI	Open	[**Open**] (= classical RH)
double-neg-RH	VI	Open	[**Open / Markov-conditional**]

A.7 Aggregate counts (for cross-check)

The cheat-sheet above contains 56 named items, matching the count in Table 2. The breakdown by stratum: 27 [**Provable**], 12 [**Conjectural**], 17 [**Open**].

B Prerequisite chain diagram

We present the prerequisite chain in textual and diagrammatic form.

B.1 Textual prerequisite chain

(P1) Volume II Parts I, II provide the cubical $\mathbb{R}_c$, $\mathbb{C}_c$ HIITs and the digit-stream final coalgebra νF_b.

(P2) Volume II Part IV provides the contractibility of $\mathrm{NNO}_{(\infty,1)}$, ensuring canonical indexing.

(P3) Volume II Part V provides GWB-style directed univalence for the discrete universe Spc.

(P4) Volume II Parts III, VI provide the foundational comparison setup and the provisional RH statement of (1).

(P5) OQ1 (Part I of Volume III) inherits (P1)–(P2) and produces the coinductive streaming witness for $\zeta(2k)$.

(P6) OQ2 (Part II) inherits (P1) and produces the constructive identity theorem for power series and the API for analytic continuation.

(P7) OQ4 (Part III) inherits (P1) and (P4) and produces the easy direction of the foundational comparison theorem.

(P8) OQ3 (Part IV) inherits (P3) and produces the finite-Segal directed univalence and the no-go observation for non-discrete types.

(P9) OQ5 (Part V) inherits (P1)–(P3) and produces $\mathbb{Q}_p$, the adele-ring HoTT colimit, and the candidate $\mathcal{E}_{\mathrm{cond}}$.

(P10) OQ6 (Part VI) inherits (P1) and (P4) and produces $\mathsf{isProp}(\mathrm{RH})$ and the certifier-decidability equivalence.

(P11) Gates 3–6 of Volume II's roadmap are populated by the OQ1–OQ6 contributions per Table 1.

(P12) The synthesis (this paper) inherits (P5)–(P11) and produces the unified six-tier stack and stratification matrix (Tables 3 and 4).

B.2 Diagrammatic prerequisite chain

$$
\begin{tikzcd}
\text{Vol II Part I} \arrow[d] & \text{Vol II Part II} \arrow[dl] \arrow[d] & \text{Vol II Part IV} \arrow[dll] \arrow[d] & \text{Vol II Part V} \arrow[d] & \text{Vol II Part III, VI} \arrow[d] \\
\text{OQ1 (Part I)} \arrow[d] \arrow[dr] & \text{OQ2 (Part II)} \arrow[d] \arrow[dr] & \text{OQ4 (Part III)} \arrow[d] & \text{OQ3 (Part IV)} \arrow[d] & \text{OQ6 (Part VI)} \arrow[d] \\
\text{Gate 3} & \text{Gate 4} & \text{Gate 5} & \text{OQ5 (Part V)} \arrow[d] & \text{Gate 6} \\
 & & & \text{Gates 5,6} &
\end{tikzcd}
\tag{6}
$$

The diagram has three layers: *Volume II inputs* (top row), *Volume III papers* (middle row), and *Gates* (bottom rows). Each Volume III paper is a node in the middle row receiving inputs from one or more Volume II nodes above and feeding into one or more Gate nodes below. The dependencies match Table 1.

C References to the six Volume III papers

Throughout this synthesis we reference the six Volume III papers via short `*-cite` bibliographic keys for theorem labels that appear in the source papers. The `-cite` suffix indicates that the label is the in-paper label (not redefined here). The reference list below collects each paper as a single bibitem, and the synthesis text refers to in-paper labels through narrative citations rather than \ref-resolved cross-references.

References

[1] M. Long. *Coinductive Witnesses for $\zeta(2k)$: A Streaming Presentation of the Euler–Bernoulli Identity in HoTT*. Volume III Part I (OQ1, coalgebraic $\zeta(2k)$), GrokRxiv:2026.05.oq1-coalgebraic-zeta-2k, 2026.

[2] Magneton Research. *OQ2 — A Cubical Library for Analytic Continuation: Identity, Monodromy, and the Booij–Mörtberg Bottleneck*. Volume III Part II (OQ2, cubical analytic continuation), GrokRxiv:2026.05.oq2-cubical-analytic-continuation, 2026.

[3] Magneton Research. *A Foundational Comparison Theorem for ζ in HoTT and ZFC*. Volume III Part III (OQ4, foundational comparison), GrokRxiv:2026.05.oq3-foundational-comparison-theorem, 2026.

[4] The YonedaAI Collaboration. *Toward Directed Univalence for Non-Discrete Types: A Partial Programme and No-Go Observations*. Volume III Part IV (OQ3, directed univalence), GrokRxiv:2026.05.oq4-directed-univalence-non-discrete, 2026.

[5] Magneton Research. *Toward an $(\infty, 1)$-Topos for Analytic Langlands: Condensed Foundations and Open Obstructions*. Volume III Part V (OQ5, analytic Langlands), GrokRxiv:2026.05.oq5-infinity-topos-analytic-langlands, 2026.

[6] RH *as a HoTT Proposition: Decidability, Modal Status, and Constructive Certification*. Volume III Part VI (OQ6, RH as proposition), GrokRxiv:2026.05.oq6-rh-as-hott-proposition, 2026.

[7] M. Long. *Volume II Synthesis: HoTT Foundations of Mathematics*. YonedaAI Collaboration, 2026.

[8] The YonedaAI Collaboration. *Volume II concluding survey: six open questions for Volume III*. GrokRxiv:2026.05.next-steps, 2026.

[9] *Volume III Knowledge Base*. YonedaAI internal document, 2026.

[10] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013. arXiv:1308.0729.

[11] A. B. Booij. *Analysis in Univalent Type Theory*. PhD thesis, University of Birmingham, 2020.

[12] C. Cohen, T. Coquand, S. Huber, A. Mörtberg. "Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom." LIPIcs 69 (TYPES 2015), 2016. arXiv:1611.02108.

[13] A. Vezzosi, A. Mörtberg, A. Abel. "Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types." Proc. ACM Program. Lang. 3 (ICFP), 2019.

[14] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer, 1976.

[15] J. J. M. M. Rutten. "Universal coalgebra: a theory of systems." *Theoretical Computer Science*, 249(1):3–80, 2000.

[16] D. Bailey, P. Borwein, S. Plouffe. "On the rapid computation of various polylogarithmic constants." *Mathematics of Computation*, 66(218):903–913, 1997.

[17] E. Bishop, D. S. Bridges. *Constructive Analysis*. Grundlehren der Mathematischen Wissenschaften 279. Springer, 1985.

[18] D. S. Bridges, F. Richman. *Varieties of Constructive Mathematics*. London Mathematical Society Lecture Note Series 97. Cambridge University Press, 1987.

[19] D. Loeffler, M. Stoll. "Formalizing zeta and L-functions in Lean." *Annals of Formalized Mathematics* 1, 2025. arXiv:2503.00959.

[20] D. Gratzer, J. Weinberger, U. Buchholtz. "Directed univalence in simplicial homotopy type theory." arXiv:2407.09146, 2024.

[21] E. Riehl, M. Shulman. "A type theory for synthetic ∞-categories." *Higher Structures* 1(1):116–193, 2017. arXiv:1705.07442.

[22] D. Clausen, P. Scholze. *Lectures on Condensed Mathematics*. University of Bonn lecture notes, 2019.

[23] D. Clausen, P. Scholze. *Analytic Stacks*. MPIM Bonn + IHES lecture course, 2023.

[24] D. Gaitsgory, S. Raskin, et al. "Proof of the geometric Langlands conjecture." Five-paper series, arXiv:2405.03599 and sequels, 2024.

[25] N. Rasekh. "A theory of elementary higher toposes." arXiv:1805.03805, 2018.

[26] N. Rasekh. "Every elementary higher topos has a natural number object." *Theory and Applications of Categories* 37(13):337–377, 2021.

[27] M. Shulman. "All $(\infty, 1)$-toposes have strict univalent universes." arXiv:1904.07004, 2019.

[28] U. Schreiber. "Differential cohomology in a cohesive infinity-topos." arXiv:1310.7930, 2013.

[29] M. Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory." *Mathematical Structures in Computer Science* 28(6):856–941, 2018.

[30] C. McLarty. "Exploring Categorical Structuralism." *Philosophia Mathematica* 12(1):37–53, 2004.

[31] R. Mines, F. Richman, W. Ruitenburg. *A Course in Constructive Algebra*. Universitext, Springer, 1988.

[32] B. Riemann. "Über die Anzahl der Primzahlen unter einer gegebenen Grösse." *Monatsberichte der Berliner Akademie*, 1859.

[33] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function*. 2nd ed., Oxford University Press, 1986.

[34] H. M. Edwards. *Riemann's Zeta Function*. Academic Press, 1974.

[35] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies 170. Princeton University Press, 2009.

[36] J. Lurie. *Higher Algebra*. Preprint, 2017.

[37] C. Barwick, P. Haine. "Pyknotic objects, I." arXiv:1904.09966, 2019.

[38] D.-C. Cisinski, H. K. Nguyen. "The universal cocartesian fibration." arXiv:2210.08945, 2022.

[39] E. Riehl. "Synthetic perspectives on spaces and categories." arXiv:2510.15795, 2025.

[40] E. Riehl, D. Verity. *Elements of ∞-Category Theory*. Cambridge Studies in Advanced Mathematics 194, Cambridge University Press, 2022.

[41] J. Tate. "Fourier analysis in number fields and Hecke's zeta-functions." In: Algebraic Number Theory (Cassels and Fröhlich eds.), Academic Press, 1967.

[42] R. Langlands. "Problems in the theory of automorphic forms." Lectures in Modern Analysis and Applications III, Lecture Notes in Mathematics 170, Springer, 1970.

[43] L. Henkin. "The completeness of the first-order functional calculus." *Journal of Symbolic Logic* 14(3):159–166, 1949.

[44] J.-L. Krivine. *Lambda-calcul: types et modèles*. Masson, 1996.

[45] P. Aczel. "The type theoretic interpretation of constructive set theory." In: Logic Colloquium '77, North-Holland, 1978.

[46] S. Mac Lane, I. Moerdijk. *Sheaves in Geometry and Logic*. Universitext, Springer, 1992.

[47] V. Voevodsky. "Univalent foundations." Lecture at the Institute for Advanced Study, 2014.

[48] J. E. Bergner. *The Homotopy Theory of $(\infty, 1)$-Categories*. London Mathematical Society Student Texts 90, Cambridge University Press, 2018.

[49] N. Kudasov, E. Riehl, J. Weinberger. *rzk: an experimental proof assistant for synthetic ∞-categories*. Software, 2026.

[50] A. Granville, G. Martin. "Prime number races." *American Mathematical Monthly* 113(1):1–33, 2006.

[51] J. Gibbons. "Unbounded spigot algorithms for the digits of π." *American Mathematical Monthly* 113(4):318–328, 2006.

[52] K. Weihrauch. *Computable Analysis: An Introduction*. Springer, 2000.

[53] J. Bernstein, A. Zelevinsky. "Representations of the group $GL(n, F)$, where F is a non-Archimedean local field." *Russian Math. Surveys* 31(3):1–68, 1976.

[54] E. Specker. "Nicht konstruktiv beweisbare Sätze der Analysis." *Journal of Symbolic Logic* 14(3):145–158, 1949.

[55] G. Kreisel. "On weak completeness of intuitionistic predicate logic." *Journal of Symbolic Logic* 27:139–158, 1962.

[56] W. Veldman. "An intuitionistic completeness theorem for intuitionistic predicate logic." *Journal of Symbolic Logic* 41:159–166, 1976.

[57] E. Palmgren. "Categories with families and first-order logic with dependent sorts." arXiv:1605.01586, 2016.

[58] M. Makkai. "First-order logic with dependent sorts, with applications to category theory." McGill preprint, 1995.

[59] B. Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observations.* Cambridge Tracts in Theoretical Computer Science 59, Cambridge University Press, 2016.

[60] M. Abbott, T. Altenkirch, N. Ghani. "Containers: Constructing strictly positive types." *Theoretical Computer Science* 342(1):3–27, 2005.

[61] U. Buchholtz, D. Gratzer. "Cocompleteness in simplicial homotopy type theory." TYPES 2025 preprint, 2025.