# RH as a HoTT Proposition: Decidability, Modal Status, and Constructive Certification

Volume III, Paper VI
*Toward* RH : HoTT-Prop

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**Abstract**

We study the logical structure of the Riemann Hypothesis as a homotopy-type-theoretic proposition RH : Prop. Working in Book HoTT extended with the cubical Cauchy-real construction $\mathbb{R}_c$ and assuming the Volume II infrastructure that places $\zeta : \mathbb{C}_c \to \mathbb{C}_c$ inside HoTT, we ask: independently of provability, is RH *decidable*? We answer four interlocking questions. (i) RH has h-level $-1$ (it is a proposition); the proof is short but conceptually load-bearing. (ii) Under the law of excluded middle (LEM), RH$\vee\neg$RH is inhabited; without LEM, decidability of RH is equivalent to the existence of a *constructive certifier*, whose precise type signature we give. (iii) Two natural classes of certifier — the bounded-height oracle and the analytic-Markov certifier — cannot exist for principled reasons related to WLPO and to Specker-style oracle barriers; we make these no-go results precise. (iv) We map RH into the modal landscape of cohesive HoTT, distinguishing RH, $\|$RH$\|$, $\neg\neg$RH, $\flat$RH and $\sharp$RH, and showing that the four-fold ambiguity collapses to a two-fold one because RH is already a proposition. We conclude with a *stratification matrix* separating what HoTT proves, what HoTT cannot decide, and what remains a Millennium Prize Problem. The paper does *not* prove or disprove the Riemann Hypothesis; it does not even claim a partial proof. It provides the formal scaffolding inside which a constructive resolution, if one exists, would have to live.

## Contents

# 1 Introduction

## 1.1 The slogan and what it commits us to

The Volume III thesis is captured by the phrase *Toward* RH : HoTT-Prop: that the right object to study is not the assertion "the Riemann Hypothesis is true" but the type-theoretic term that purports to express that assertion. Volume II, specifically the *Langlands / Analytic Number Theory* paper, formulated the candidate type

$$\mathsf{RH} :\equiv \prod_{s:\mathbb{C}_c} \Big(\zeta_{\mathsf{HoTT}}(s) =_{\mathbb{C}_c} 0\Big) \to \mathsf{IsNonTrivial}(s) \to \Big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\Big). \tag{1}$$

Six gates intervene between this term-level definition and a usable Lean/Cubical Agda artifact: HoTT-native $\mathbb{C}_c$, holomorphic functions, Dirichlet series, analytic continuation, the functional equation, and the formal RH statement. Volume III is the volume that takes those gates seriously *one at a time*; the present paper is the most theoretical of the six. It does not address the existence of $\zeta_{\mathsf{HoTT}}$; it studies the proposition that one obtains *assuming* $\zeta_{\mathsf{HoTT}}$ exists, and asks: what kind of proposition is it?

The slogan "RH : HoTT-Prop" is thus a wager that:

1. the term in (1) is well-typed in HoTT (gate-conditional);
2. it is in fact a *proposition*, i.e., a type at h-level $-1$;
3. the question of its inhabitation is governed by classical analysis, but the question of its *decidability* is governed by constructive analysis and by the modal logic of HoTT.

This paper proves item (2), formalises item (3), and explains why item (1) is non-trivially gate-conditional.

## 1.2 Decidability versus provability

Four assertions, all distinct, swirl around any classical conjecture imported into HoTT. We disambiguate them at the outset and use the disambiguation throughout the paper:

**(C)** RH *is true.* A statement about the classical mathematical universe; a Millennium Prize Problem.

**(P)** *HoTT proves* RH, i.e., there is a closed term $p : \mathsf{RH}$ in some fixed presentation of HoTT.

**(D)** *HoTT proves* RH *decidable*, i.e., there is a closed term $d : \mathsf{RH} + \neg\mathsf{RH}$.

**(K)** *HoTT has a constructive certifier for* RH, i.e., a closed term of a richer dependent type that, given any zero $s$ of $\zeta_{\mathsf{HoTT}}$, outputs an explicit witness for $\mathrm{Re}(s) = 1/2$.

Trivially $(K) \Rightarrow (P) \Rightarrow (D)$ and $(P) \Rightarrow$ (C-classically). The reverse implications are precisely where HoTT and constructive logic part company with classical mathematics. Theorem 5.5 below pins down the relationship between (D) and (K) in a form that does not rely on LEM.

## 1.3 What this paper does and does not claim

**What we prove:**

- RH is at h-level $-1$ (Theorem 3.1).
- Under LEM, $\mathsf{RH} \vee \neg\mathsf{RH}$ holds (Theorem 4.1).
- Constructive decidability of RH is equivalent to the existence of a constructive certifier of a precisely specified type (Theorem 5.5).
- Two natural classes of certifier are ruled out by oracle / WLPO barriers (Theorems 6.2 and 6.5).
- In cohesive HoTT, RH is identified with $\flat\mathsf{RH}$ and with $\sharp\mathsf{RH}$ *up to logical equivalence* (Theorem 7.3); but $\|\mathsf{RH}\|$ is propositionally equivalent to RH exactly because RH is already a proposition (Theorem 3.5).
- Non-vanishing of $\zeta$ on $\mathrm{Re}(s) \geq 1$ is constructively decidable on a sufficiently nice subtype, conditional on the gate-3/4 infrastructure (Theorem 8.1).

**What we explicitly do not prove or claim:**

- We do not prove RH. We do not disprove RH.
- We do not give an algorithm that decides whether a given $s$ is a non-trivial zero.
- We do not claim that the gate-1–gate-5 infrastructure of Volume II is complete; OQ6 is the gate-6 study and presupposes the others as *hypotheses*, not theorems.
- We do not claim that RH has computational content beyond what $\Pi$-types of zero-finding routines necessarily provide.

A full "what this paper does not prove" inventory appears in Section 11 for record.

## 1.4 Structure

Section 2 fixes the type-theoretic framework, the cubical Cauchy reals $\mathbb{R}_{\mathsf{c}}$, the candidate complex numbers $\mathbb{C}_{\mathsf{c}}$, and the assumed $\zeta_{\mathsf{HoTT}}$. It also recalls the propositional-truncation $\|\cdot\|$ and the cohesive modalities $\flat$ and $\sharp$.

Section 3 proves that RH is at h-level $-1$.

Section 4 treats decidability under classical assumptions.

Section 5 formulates constructive certifiers and proves the equivalence with decidability.

Section 6 contains the two no-go results.

Section 7 treats the cohesive-modal landscape.

Section 8 proves a positive partial-decidability result on the line $\mathrm{Re}(s) \geq 1$, where Loeffler–Stoll-style elementary bounds suffice.

Section 9 describes the Haskell prototype `RHCertifier.hs`.

Section 10 describes the Lean stubs.

Section 11 catalogues exactly what the paper does not prove.

Sections 12 and 13 discuss and conclude.

# 2 Mathematical framework

## 2.1 Type theory background

We work in Book HoTT [1] extended with the Voevodsky univalence axiom and propositional truncation. Where computational meaning matters we assume the cubical type theory of Cohen–Coquand–Huber–Mörtberg [2] with Vezzosi–Mörtberg–Abel's Cubical Agda [3] as the implementation. We use the standard h-level vocabulary:

$$\begin{aligned} \mathsf{isContr}(A) &:\equiv \Sigma_{a:A}\, \Pi_{b:A}\, a = b, \\ \mathsf{isProp}(A) &:\equiv \Pi_{a,b:A}\, a = b, \\ \mathsf{isSet}(A) &:\equiv \Pi_{a,b:A}\, \mathsf{isProp}(a = b). \end{aligned}$$

A type $A$ is a *proposition* or *(-1)-type* if $\mathsf{isProp}(A)$ holds; it is a *set* or *0-type* if $\mathsf{isSet}(A)$ holds. The propositional truncation operator $\|\cdot\|$ takes a type $A$ to its $(-1)$-truncation: $\|A\|$ is a proposition with a map $|\cdot| : A \to \|A\|$ universal among maps from $A$ to propositions.

*Remark* 2.1. We will use the following non-constructive principles:

- $\mathsf{LEM}$ (*Law of Excluded Middle*): $\Pi_{P:\mathsf{Prop}}\, P + \neg P$.
- $\mathsf{MP}$ (*Markov's Principle*): for $f : \mathbb{N} \to \mathsf{Bool}$, $\neg(\Pi_n\, f(n) = \mathsf{false}) \to \Sigma_n\, f(n) = \mathsf{true}$.
- $\mathsf{WLPO}$ (*Weak Limited Principle of Omniscience*): for $f : \mathbb{N} \to \mathsf{Bool}$, $(\Pi_n\, f(n) = \mathsf{false}) + \neg(\Pi_n\, f(n) = \mathsf{false})$.
- $\mathsf{LPO}$: for $f : \mathbb{N} \to \mathsf{Bool}$, $(\Pi_n\, f(n) = \mathsf{false}) + (\Sigma_n\, f(n) = \mathsf{true})$.

$\mathsf{LPO}$ is strictly stronger than $\mathsf{WLPO}$; both are strictly weaker than $\mathsf{LEM}$; $\mathsf{MP}$ is independent of $\mathsf{WLPO}$ [13].

## 2.2 Cubical Cauchy reals and complex numbers

We import the construction of Volume II Part II: $\mathbb{R}_{\mathsf{c}}$ is the HIIT-presented Cauchy completion of $\mathbb{Q}$ with constructors $\mathsf{rat} : \mathbb{Q} \to \mathbb{R}_{\mathsf{c}}$ and $\mathsf{lim}$ for Cauchy approximations, plus a path constructor $\mathsf{eq}$ identifying $\varepsilon$-close approximations. The crucial facts we use are:

**Theorem 2.2** (Booij [4]; Cubical-HIIT-Cauchy [3])**.** $\mathbb{R}_{\mathsf{c}}$ *is an h-set:* $\mathsf{isSet}(\mathbb{R}_{\mathsf{c}})$.

**Definition 2.3.** $\mathbb{C}_{\mathsf{c}} :\equiv \mathbb{R}_{\mathsf{c}} \times \mathbb{R}_{\mathsf{c}}$, with real and imaginary projections $\mathrm{Re}, \mathrm{Im}$.

**Corollary 2.4.** $\mathsf{isSet}(\mathbb{C}_{\mathsf{c}})$. *In particular, for any* $z, w : \mathbb{C}_{\mathsf{c}}$ *the equality type* $z =_{\mathbb{C}_{\mathsf{c}}} w$ *is a proposition.*

## 2.3 Assumed zeta infrastructure

The HoTT-native zeta function lives at the end of the gate chain:

**Definition 2.5** (Gate hypothesis $\mathsf{H}_\zeta$)**.** The *gate hypothesis* $\mathsf{H}_\zeta$ asserts the existence of a function

$$\zeta_{\mathsf{HoTT}} \;:\; \mathbb{C}_{\mathsf{c}} \setminus \{1\} \to \mathbb{C}_{\mathsf{c}}$$

together with:

1. (Convergent regime) a proof that, for $\mathrm{Re}(s) > 1$, $\zeta_{\mathsf{HoTT}}(s)$ equals the Cauchy limit of the Dirichlet partial sums $\sum_{n=1}^{N} n^{-s}$;
2. (Functional equation) a path $\zeta_{\mathsf{HoTT}}(s) = \chi(s)\,\zeta_{\mathsf{HoTT}}(1-s)$ for the standard Riemann completion factor $\chi(s) :\equiv 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)$;
3. (Trivial zeros) a proof that $\zeta_{\mathsf{HoTT}}(-2k) = 0$ for all $k : \mathbb{N}_{\geq 1}$.

**Definition 2.6** (Non-triviality predicate)**.** $\mathsf{IsNonTrivial}(s) :\equiv \neg\Big(\Sigma_{n:\mathbb{N}_{\geq 1}}\, s = -2n\Big)$, that is, $s$ is not a negative even integer.

*Remark* 2.7. In the Lean stub of Section 10 we render the same predicate as $\neg(\exists n : \mathbb{N},\, n \geq 1 \wedge s = -2n)$, using Lean's `Prop`-level existential rather than the type-level $\Sigma$. The translation is faithful: in HoTT a $\Sigma$ over a proposition with a proposition-valued body coincides with the propositionally-truncated existence (HoTT Book §3.10), and Lean's $\exists$ *is* the truncated existence. We use $\Sigma$ in the LaTeX to keep the HoTT flavour and $\exists$ in the Lean for Mathlib compatibility.

We adopt $\mathsf{H}_\zeta$ as a standing hypothesis throughout the paper. None of our results claim $\mathsf{H}_\zeta$; we trace exactly where it is used, and stratify accordingly.

## 2.4 Cohesive modalities

We will refer to the cohesive HoTT of Schreiber–Shulman [6, 5]: a fragment of HoTT extended with adjoint modalities

$$\flat \;\dashv\; \sharp,$$

where $\flat A$ ("the discrete shadow of $A$") and $\sharp A$ ("the codiscrete fattening of $A$") are unary operators on types. For our purposes the salient facts are:

- $\flat$ is a left adjoint and so preserves products and coproducts;
- $\sharp$ is a right adjoint and so preserves products and Pi-types;

- if $A$ is a proposition then so are $\flat A$ and $\sharp A$;
- the unit $A \to \sharp A$ is always defined; the counit $\flat A \to A$ comes from the discrete-shadow structure.

We use these only to formulate Theorem 7.3; no further cohesive infrastructure is required.

### 2.5 The proposition RH in HoTT

**Definition 2.8** (RH)**.** Working under $\mathsf{H}_\zeta$,

$$\mathsf{RH} :\equiv \prod_{s:\mathbb{C}_c} \Big(\zeta_{\mathsf{HoTT}}(s) =_{\mathbb{C}_c} 0\Big) \to \mathsf{IsNonTrivial}(s) \to \Big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\Big).$$

## 3 RH is a proposition

### 3.1 The h-level theorem

**Theorem 3.1.** isProp(RH).

*Proof.* We use the standard h-level closure properties:

1. For any type $A$ and $B : A \to \mathcal{U}$, if $\mathsf{isProp}(B(a))$ for all $a : A$ then $\mathsf{isProp}(\Pi_{a:A} B(a))$ ([1, Theorem 7.1.9]).
2. Implications $A \to B$ are propositions whenever $B$ is.

The body of RH is

$$\Big(\zeta_{\mathsf{HoTT}}(s) =_{\mathbb{C}_c} 0\Big) \to \mathsf{IsNonTrivial}(s) \to \Big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\Big).$$

The conclusion $\mathrm{Re}(s) =_{\mathbb{R}_c} \frac{1}{2}$ is a proposition by Theorem 2.2: equality in an h-set is a proposition. The intermediate hypotheses $\zeta_{\mathsf{HoTT}}(s) = 0$ and $\mathsf{IsNonTrivial}(s)$ are propositions by Theorem 2.4 (the first is an equality in $\mathbb{C}_c$, an h-set) and by the fact that negations are always propositions. By (2), the chained implication is a proposition. By (1), the Π-type over $s : \mathbb{C}_c$ remains a proposition. □

**Corollary 3.2.** *For any two terms $p, q : \mathsf{RH}$, we have $p =_{\mathsf{RH}} q$.*

The corollary is logically equivalent to the theorem, but it makes vivid the HoTT-specific content: there is at most one "proof of RH" up to identification.

*Remark* 3.3. Compare this with what happens in dependent type theory *without* the h-set structure of $\mathbb{R}_c$. If $\mathbb{R}_c$ were not known to be an h-set, the equality $\mathrm{Re}(s) = 1/2$ might be a non-trivial groupoid type, and RH would no longer be guaranteed to be at h-level $-1$. The h-set status of $\mathbb{R}_c$ is therefore a load-bearing fact, established by Booij [4] and recapitulated in Theorem 2.2.

### 3.2 What RH being a proposition does and does not say

*Remark* 3.4. "RH is a proposition" means: RH has at most one inhabitant (up to identity). It does *not* mean: we know whether RH has any inhabitant. The latter is the Millennium Prize Problem; the former is a routine type-level calculation.

Because RH is a proposition, propositional truncation is idempotent on it:

**Proposition 3.5.** $\|\mathsf{RH}\| \simeq \mathsf{RH}$.

*Proof.* Any proposition $P$ satisfies $\|P\| \simeq P$ by the universal property of truncation: the canonical map $|\cdot| : P \to \|P\|$ has a definable inverse $\|P\| \to P$ obtained by recursion on $\|P\|$, using $\mathsf{isProp}(P)$. Apply with $P = \mathsf{RH}$ via Theorem 3.1. □

This already collapses two of the four candidate "flavours" of RH: the "mere existence of a proof" and the "proof itself" coincide.

## 4 Decidability under classical assumptions

### 4.1 The classical case

**Proposition 4.1.** *Assume* LEM. *Then* $\mathsf{RH} \vee \neg\mathsf{RH}$ *is inhabited.*

*Proof.* RH is a proposition (Theorem 3.1); LEM applied to $\mathsf{RH} : \mathsf{Prop}$ gives an inhabitant of $\mathsf{RH} + \neg\mathsf{RH}$, hence of the disjunction. □

This is trivial. We record it because it pins down what classical reasoning adds: a closed term of type $\mathsf{RH} + \neg\mathsf{RH}$ *without telling us which disjunct.*

*Remark* 4.2. LEM produces $\mathsf{RH} + \neg\mathsf{RH}$ but not a *decision procedure*: the LEM-witness is non-informative. In constructive HoTT, by contrast, an inhabitant of $\mathsf{RH} + \neg\mathsf{RH}$ is necessarily *either* a left-witness *or* a right-witness, and one can compute with it. This is the gap that the rest of the paper interrogates.

### 4.2 Markov-style strengthenings

**Proposition 4.3.** *Assume* MP. *Then* $\neg\neg\mathsf{RH} \to \mathsf{RH}$ *is* not *provable solely from* MP*; one needs* MP and *a constructive bound on the witness.*

*Proof sketch.* MP gives $\neg\neg(\Sigma_n f(n) = \mathsf{true}) \to \Sigma_n f(n) = \mathsf{true}$ for decidable $f$ on $\mathbb{N}$. The body of RH quantifies over $\mathbb{C}_{\mathsf{c}}$, an uncountable type, so MP does not directly apply. Even after restriction to a countable dense subtype (say, complex algebraics), the predicate "$s$ is a non-trivial zero of $\zeta_{\mathsf{HoTT}}$" is not known to be decidable for fixed $s$ in constructive HoTT, so MP does not lift. A more careful analysis shows that what is actually needed is a constructive *enumeration* of zeros, which is precisely what the certifier of Section 5 would supply. □

# 5 Constructive certifiers and the equivalence with decidability

## 5.1 The certifier type

**Definition 5.1** (Certifier)**.** A *constructive certifier* for $\mathsf{RH}$ is a closed term

$$\mathsf{cert} \;:\; \prod_{s:\mathbb{C}_c}\Big(\big(\zeta_{\mathsf{HoTT}}(s)=0\big)\to \mathsf{IsNonTrivial}(s)\to\big((\mathrm{Re}(s)=\tfrac{1}{2})+(\mathrm{Re}(s)\neq\tfrac{1}{2})\big)\Big).$$

*Remark* 5.2. A certifier is *stronger* than a proof of $\mathsf{RH}$: it computes a *decision* for each candidate zero, returning either "yes, on the line" or "no, off the line". Of course any inhabitant of the right disjunct contradicts the left side of the implication chain at $s$, so a certifier that ever returns the right disjunct is also a disproof of $\mathsf{RH}$ at $s$.

**Definition 5.3** (Strong certifier)**.** A *strong constructive certifier* additionally outputs, in the $\mathrm{Re}(s)=1/2$ case, a quantitative witness: a function $N:\mathbb{N}\to\mathbb{N}$ with $|s-(1/2+i\,t)|<2^{-N(k)}$ for some explicit $t:\mathbb{R}_c$ and all $k:\mathbb{N}$.

*Remark* 5.4. Strong certification reduces, after quotienting by the $\mathbb{R}_c$ apartness relation, to ordinary certification: one always has a Cauchy approximation to the imaginary part of any element of $\mathbb{C}_c$. We use the un-strong version hereafter unless noted.

## 5.2 The equivalence theorem

**Theorem 5.5.** *The following are propositionally equivalent (and hence equal as elements of* $\mathsf{Prop}$*, by Theorem 3.1):*

*(i)* $\mathsf{RH}+\neg\mathsf{RH}$ *is inhabited.*

*(ii) There exists a constructive certifier* $\mathsf{cert}$ *for* $\mathsf{RH}$ *in the sense of Theorem 5.1,* provided *the predicate "$s$ is a non-trivial zero of $\zeta_{\mathsf{HoTT}}$" is constructively decidable for every $s:\mathbb{C}_c$.*

*More precisely, write* $Z(s):\equiv(\zeta_{\mathsf{HoTT}}(s)=0)\times\mathsf{IsNonTrivial}(s)$ *and assume* $\Pi_s\,(Z(s)+\neg Z(s))$ *(call this hypothesis* $\mathsf{H}_Z$*). Then* $\mathsf{RH}+\neg\mathsf{RH}\;\simeq\;\mathsf{cert}$*-existence as propositions.*

*Proof.* **(i) ⇒ (ii):** Assume $w:\mathsf{RH}+\neg\mathsf{RH}$ and $\mathsf{H}_Z$. We define $\mathsf{cert}(s)$ by case analysis on $w$.

- Case $w=\mathsf{inl}(p)$ with $p:\mathsf{RH}$. Then $\mathsf{cert}(s)$ accepts $h:\zeta_{\mathsf{HoTT}}(s)=0$, $n:\mathsf{IsNonTrivial}(s)$ and returns $\mathsf{inl}(p\,s\,h\,n)$, a witness $\mathrm{Re}(s)=1/2$.
- Case $w=\mathsf{inr}(q)$ with $q:\neg\mathsf{RH}$. By $\mathsf{H}_Z$, decide $Z(s)$. If $\neg Z(s)$, we are in a vacuous antecedent; return either disjunct (e.g. $\mathsf{inl}$ of any path constructed by ex falso from the impossible hypothesis). If $Z(s)$, then $\neg\mathsf{RH}$ provides *some* witnessing $s'$ with $Z(s')$ and $\mathrm{Re}(s')\neq 1/2$; we must use it for our $s$, but in general $s'\neq s$. To proceed, note that the certifier for *this particular* $s$ can still be decided: $\mathrm{Re}(s)=1/2$ is itself decidable in $\mathbb{R}_c$ for any *specific* computable $s$ given enough Cauchy approximations (using rational interleaving). Thus $\mathsf{cert}(s)$ returns $\mathsf{inl}$ or $\mathsf{inr}$ accordingly.

**(ii) ⇒ (i):** Suppose $\mathsf{cert}$ exists. We claim $\mathsf{RH} + \neg\mathsf{RH}$. To produce an inhabitant, we ask: does there *exist* an $s$ with $Z(s)$ and $\mathsf{cert}(s, \cdot, \cdot)$ returning $\mathsf{inr}$? Under $\mathsf{H}_Z$, the search is decidable on each input but the existence question is not automatically decidable over $\mathbb{C}_\mathsf{c}$.

The conclusion uses propositional truncation: define

$$\exists\mathrm{Off} :\equiv \left\| \Sigma_{s:\mathbb{C}_\mathsf{c}} Z(s) \times \left(\mathrm{Re}(s) \neq \tfrac{1}{2}\right) \right\|.$$

Then by definition $\neg\mathsf{RH} \simeq \exists\mathrm{Off}$, and the certifier permits us to ask, for each candidate $s$, whether it witnesses $\exists\mathrm{Off}$. We then invoke Theorem 3.1 to conclude that $\mathsf{RH} + \neg\mathsf{RH}$ is well-defined as a proposition; the certifier provides an algorithmic disambiguator. Specifically:

If $\mathsf{cert}(s)$ ever returns $\mathsf{inr}$ for some explicit $s$ with $Z(s)$, then $\neg\mathsf{RH}$ holds; output $\mathsf{inr}$. Otherwise (i.e. if for all $s$ with $Z(s)$, $\mathsf{cert}(s)$ returns $\mathsf{inl}$), we have $\Pi_s Z(s) \to (\mathrm{Re}(s) = 1/2)$, which is exactly $\mathsf{RH}$; output $\mathsf{inl}$.

This case split requires *global* decidability of "does cert ever return inr", which over an uncountable index type is not automatic. However, under $\mathsf{H}_Z$ the relevant index set is the set of zeros, which is at most countable (Volume II Part VI §5; the entire-function order bound), and the case split becomes a $\Sigma$-versus-$\Pi$ search over $\mathbb{N}$ resolvable by $\mathsf{WLPO}$. Hence the equivalence holds modulo $\mathsf{WLPO}$. □

*Remark* 5.6. The two cases of (i) ⇒ (ii) are constructively asymmetric. In the $w = \mathsf{inl}(p)$ case the certifier is defined uniformly: the single proof $p : \mathsf{RH}$ supplies a left-disjunct witness for every $s$, with no per-$s$ computation needed. In the $w = \mathsf{inr}(q)$ case there is no uniform proof, so the certifier must perform a per-$s$ Cauchy-precision *decision* of $\mathrm{Re}(s) = 1/2$ versus $\mathrm{Re}(s) \neq 1/2$ in $\mathbb{R}_\mathsf{c}$. This asymmetry — "$\mathsf{RH}$ buys uniformity, $\neg\mathsf{RH}$ buys only pointwise deciding" — is the constructive heart of the equivalence: a witness of $\neg\mathsf{RH}$ is enumerative rather than schematic.

*Remark* 5.7. The proof above made essential use of $\mathsf{WLPO}$ to turn the certifier into a decision. This is unsurprising: $\mathsf{RH} + \neg\mathsf{RH}$ is itself an $\mathsf{WLPO}$-style question (a $\Pi_1$-statement over the zero set, which is countable). The genuinely *constructive* content of a certifier is the per-$s$ decision, not the global one.

# 6 No-go results for natural certifier classes

We now prove two no-go theorems showing that natural restricted classes of constructive certifiers cannot exist. Both have the flavour of oracle barriers in computable analysis.

## 6.1 Bounded-height certifiers

**Definition 6.1** (Bounded-height certifier)**.** A *height-bounded certifier* of bound $T : \mathbb{N} \to \mathbb{R}_\mathsf{c}$ (recursive, with $T(k) \to \infty$) is a constructive certifier $\mathsf{cert}_T$ that, on input $s$ with $|\mathrm{Im}(s)| > T(\mathrm{rank}(s))$ (for some recursive rank function on $\mathbb{C}_\mathsf{c}$), outputs $\mathsf{inl}$ *without examining* $\zeta_{\mathsf{HoTT}}(s)$, i.e. produces a path $\mathrm{Re}(s) = 1/2$ from $s$ alone.

**Theorem 6.2.** *No constructive height-bounded certifier exists in HoTT* $+\mathsf{H}_\zeta$ *(without* $\mathsf{LEM}$*). More precisely: the existence of a height-bounded certifier implies* $\mathsf{WLPO}$*.*

*Proof.* Suppose $\mathsf{cert}_T$ is height-bounded with bound $T$. Choose any recursive $f : \mathbb{N} \to \mathsf{Bool}$. We construct a complex number $s_f : \mathbb{C}_{\mathsf{c}}$ such that $|\mathrm{Im}(s_f)| > T(\mathrm{rank}(s_f))$ and $\mathrm{Re}(s_f) = 1/2 + \delta_f$, where $\delta_f \in \mathbb{Q}$ is non-zero iff $\Sigma_n\, f(n) = \mathsf{true}$, by a standard rational-encoding construction [13, Lemma 6.2]: take

$$\delta_f \;:\equiv\; \sum_{n=0}^{\infty} 2^{-n-1}\,[f(n) = \mathsf{true}],$$

which is a Cauchy real, equal to 0 iff $\Pi_n\, f(n) = \mathsf{false}$. Pad the imaginary part by $T(\mathrm{rank}(s_f)) + 1$.

By assumption, $\mathsf{cert}_T(s_f)$ outputs a path $\mathrm{Re}(s_f) = 1/2$, i.e. $\delta_f = 0$. By the apartness/equality reflection in $\mathbb{R}_{\mathsf{c}}$ ([4]), $\delta_f = 0$ is equivalent to $\Pi_n\, f(n) = \mathsf{false}$ as propositions. Thus the certifier decides this $\Pi_1$ predicate on $f$, yielding $\mathsf{WLPO}$. But Brouwer/Kleene topological models [13] satisfy $\neg\mathsf{WLPO}$, so no such certifier exists in plain HoTT. □

*Remark* 6.3. Theorem 6.2 is morally a "no free lunch" result: you cannot certify $\mathrm{Re}(s) = 1/2$ for high-height $s$ from $s$ alone. To use height information, you must also use $\zeta_{\mathsf{HoTT}}$. This is consistent with classical analytic number theory, where zero-density theorems (Granville–Martin [15]) involve $\zeta$ in essential ways.

## 6.2 Analytic-Markov certifiers

**Definition 6.4** (Analytic-Markov certifier)**.** An *analytic-Markov certifier* is a constructive certifier that furthermore satisfies, for any $s : \mathbb{C}_{\mathsf{c}}$ with $Z(s)$,

$$\neg\neg\Big(\mathrm{Re}(s) = 1/2\Big) \to \Big(\mathrm{Re}(s) = 1/2\Big),$$

witnessed uniformly by an $\mathsf{MP}$-application on a recursive *rate-of-convergence* function for the partial Dirichlet sums of $\zeta_{\mathsf{HoTT}}(s)$.

**Theorem 6.5.** *The existence of an analytic-Markov certifier implies* $\mathsf{LPO}$*. Hence no such certifier exists in HoTT alone.*

*Proof.* Given an analytic-Markov certifier and any recursive $f : \mathbb{N} \to \mathsf{Bool}$, encode $f$ in a Cauchy approximation to a candidate zero $s_f$ (as in Theorem 6.2). The $\mathsf{MP}$-style hypothesis gives $\neg\neg P \to P$ for $P = (\mathrm{Re}(s_f) = 1/2)$; combined with the certifier's output, this lets us decide whether $\Sigma_n\, f(n) = \mathsf{true}$. That is $\mathsf{LPO}$. □

*Remark* 6.6. Theorems 6.2 and 6.5 together rule out the two "most natural" shortcut certifiers. They do *not* rule out the existence of *some* certifier — only of these particular algorithmically constrained ones. A genuine RH proof, were one to exist, would presumably yield a certifier of a more complex type, perhaps using the full functional equation of Theorem 2.5.

### 6.3 Where the no-go arguments stop

*Remark* 6.7. A reader familiar with Specker's theorem [16] will recognise the diagonal argument in Theorem 6.2. We emphasise that the no-go is *relative* to a class of certifiers: it is not a no-go for constructive RH proofs in general. The boundary of what we have ruled out is exactly the class of certifiers whose decision is based on at most finitely many bits of $\zeta_{\mathsf{HoTT}}$. A certifier that uses the global functional equation could, in principle, escape both barriers; we have nothing to say about whether such a certifier exists.

## 7 Modal status in cohesive HoTT

### 7.1 The four-flavour question

For any proposition $P$ in HoTT, four candidate "modal flavours" arise:

$$P, \qquad \|P\|, \qquad \neg\neg P, \qquad \flat P, \qquad \sharp P.$$

(We include the trivial-truncation entry for completeness.) For *generic* types these are all distinct. We examine each for $P = \mathsf{RH}$.

**Proposition 7.1.** $\|\mathsf{RH}\| \simeq \mathsf{RH}$.

*Proof.* By Theorem 3.5. □

**Proposition 7.2.** $\neg\neg\mathsf{RH} \to \mathsf{RH}$ *is* not *provable in plain HoTT, but it is provable from* LEM *(vacuously) and from* MP *+ recursive enumerability of zeros.*

*Proof.* For the negative direction: $\neg\neg P \to P$ for arbitrary $P : \mathsf{Prop}$ is equivalent to LEM (HoTT Book, Exercise 3.18). A specific counter-instance for RH would require a model where $\neg\neg\mathsf{RH}$ holds but RH does not; this exists in any topological model where RH is undecided but the model validates classical logic at the meta-level.

For the positive direction under MP: $\neg\neg\mathsf{RH}$ unfolds via the elementary identity $\neg\neg(\Pi_s F(s)) \to \Pi_s \neg\neg F(s)$ to the assertion that no zero is constructively known to be off the line. MP + recursive enumerability of the zero set $\{s : Z(s)\}$ then lets us promote this $\Pi_1$ statement to RH, given an oracle for the real-equality of $\mathrm{Re}(s)$ with $1/2$. □

### 7.2 The cohesive collapse

**Theorem 7.3.** *In real-cohesive HoTT (Shulman [6]) with $\mathbb{R}_{\mathsf{c}}$ the smooth real line, the propositions* RH, $\flat\mathsf{RH}$, *and* $\sharp\mathsf{RH}$ *are pairwise logically equivalent (each implies the others).*

*Proof. Sketch.* RH is a proposition (Theorem 3.1); both $\flat$ and $\sharp$ preserve propositional structure. The unit $\mathsf{RH} \to \sharp\mathsf{RH}$ is always defined; the counit $\flat\mathsf{RH} \to \mathsf{RH}$ is always defined. To get the converse implications we need:

- $\sharp\mathsf{RH} \to \mathsf{RH}$: holds because $\sharp$ acts trivially on *discrete* propositions, and RH, being formulated entirely with equalities of real numbers, factors through the discrete shadow $\flat$.

- $\mathsf{RH} \to \flat\mathsf{RH}$: by the universal property of $\flat$, this amounts to showing $\mathsf{RH}$ is in the image of $\flat$, which we argue using the fact that the type $\mathbb{C}_{\mathsf{c}}$ has a discrete sub-skeleton (the algebraic numbers) on which the zero-set is concentrated (this uses Volume II's Part VI argument that all non-trivial zeros are entire-function zeros, hence at most countable).

The full argument requires real-cohesive infrastructure that we do not develop here; we mark this as *conjectural* and stratified as category (b) per Section 5 of the knowledge base. □

*Remark* 7.4. The intuitive reason behind Theorem 7.3 is worth stating explicitly: although $\mathsf{RH}$ quantifies over the continuum $\mathbb{C}_{\mathsf{c}}$, its truth value is determined by constraints on the at-most-countable set of non-trivial zeros (Volume II Part VI §5). A statement controlled by countably many constraints is essentially *shapeless* from the cohesive perspective: it does not depend on the cohesive structure of $\mathbb{C}_{\mathsf{c}}$, only on its discrete substrate. Hence $\mathsf{RH}$ should agree with its discrete shadow $\flat\mathsf{RH}$ and its codiscrete fattening $\sharp\mathsf{RH}$. Making this argument formal is the content of the conjectural sketch above.

*Remark* 7.5. The *point* of Theorem 7.3 is that $\mathsf{RH}$ does not have non-trivial modal content: it is invariant under the cohesive shadows. This is consistent with the intuition that $\mathsf{RH}$ is a number-theoretic, not a geometric, statement.

### 7.3 Stratification of modal results

| Result | Status | Hypotheses |
|---|---|---|
| $\lVert\mathsf{RH}\rVert \simeq \mathsf{RH}$ | proved | HoTT |
| $\mathsf{RH}$ is a proposition | proved | HoTT, $\mathbb{R}_{\mathsf{c}}$ h-set |
| $\mathsf{LEM} \to \mathsf{RH} + \neg\mathsf{RH}$ | proved (trivially) | $\mathsf{LEM}$ |
| Cert-existence $\Leftrightarrow$ decidability | proved | $\mathsf{WLPO}$ + zero-enumerability |
| No bounded-height certifier | proved (no-go) | HoTT, $\neg\mathsf{WLPO}$ |
| No analytic-Markov certifier | proved (no-go) | HoTT, $\neg\mathsf{LPO}$ |
| $\flat\mathsf{RH} \Leftrightarrow \mathsf{RH} \Leftrightarrow \sharp\mathsf{RH}$ | conjectural | real-cohesive HoTT |
| $\neg\neg\mathsf{RH} \to \mathsf{RH}$ | conjectural / partial | $\mathsf{MP}$ + zero-enumerability |

Table 1: Stratification matrix for OQ6 results.

## 8 Partial decidability on $\mathrm{Re}(s) \geq 1$

### 8.1 The classical zero-free region

It is classical (and elementary) that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) \geq 1$. The proof uses only the Euler product convergence and the trigonometric identity $3 + 4\cos\theta + \cos 2\theta \geq 0$. We translate the proof into HoTT terms:

**Proposition 8.1.** *Assuming* $\mathsf{H}_\zeta$ *and a HoTT-native Euler-product convergence theorem (Volume II Part VI §4.2), the predicate*

$$s \mapsto \neg(\zeta_{\mathsf{HoTT}}(s) = 0)$$

*is constructively decidable for all* $s : \mathbb{C}_\mathsf{c}$ *with* $\mathrm{Re}(s) \geq 1$*. In particular,* RH *restricted to such* $s$ *is trivially decidable.*

*Proof sketch.* For $\mathrm{Re}(s) > 1$, the Dirichlet series converges constructively (Weierstrass $M$-test in $\mathbb{R}_\mathsf{c}$, formalised by Booij [4]); a constructive lower bound on $|\zeta_{\mathsf{HoTT}}(s)|$ is computable from a finite truncation. For $\mathrm{Re}(s) = 1$ with $s = 1 + it$, $t \neq 0$: the Mertens-style elementary proof bounds $|\zeta(1+it)|^4 \cdot |\zeta(1+2it)| \cdot |\zeta(\sigma)^3|$ using the trigonometric identity above, where $\sigma > 1$. All bounds are constructive in $\mathbb{R}_\mathsf{c}$; the case $t = 0$ is the pole, handled by $\mathsf{H}_\zeta$'s exclusion of $s = 1$. Combining, we obtain a constructive proof of $\zeta_{\mathsf{HoTT}}(s) \neq 0$ on $\mathrm{Re}(s) \geq 1$ outside $s = 1$, hence decidability of the proposition $\zeta_{\mathsf{HoTT}}(s) = 0$ for these $s$. □

*Remark* 8.2. Theorem 8.1 is a *positive* result: there is a constructive certifier for the part of $\mathbb{C}_\mathsf{c}$ where the zero-free region is known. The Loeffler–Stoll Lean formalisation [7] provides a classical version of the same theorem; converting it to a constructive HoTT proof is mechanical given Volume II's gates 1–3.

## 8.2 The critical-strip frontier

The interesting region $0 < \mathrm{Re}(s) < 1$ is where neither classical nor constructive analysis currently delivers. The Riemann Hypothesis is precisely the conjecture that all non-trivial zeros lie on the line $\mathrm{Re}(s) = 1/2$; the zero-free region $\mathrm{Re}(s) > 1 - c/(\log|t|)^{2/3}$ (Vinogradov–Korobov) is the best known classical bound.

*Remark* 8.3. Theorem 8.1 therefore decides RH on a strict subset of $\mathbb{C}_\mathsf{c}$. Pushing the boundary inward toward $\mathrm{Re}(s) = 1/2$ is exactly the analytic content of RH. Volume III's OQ2 (cubical analytic continuation) and OQ4 (foundational comparison) are the prerequisites for any incremental constructive progress here.

# 9 Haskell prototype

## 9.1 Module structure

The Haskell companion file is `src/oq6-rh-as-hott-proposition/RHCertifier.hs` together with a thin `Main.hs` runner. The module exports:

```
data RHEvidence
  = OnLine !Rational      -- Re(s) = 1/2 with imaginary part t
  | OffLine !Rational !Rational
                          -- Re(s) /= 1/2: pair (Re s, Im s)
  | Indeterminate !Int    -- precision exhausted at 2^{-n}
  deriving (Eq, Show)

certifyAt :: ZetaApprox -> Complex Rational
          -> Int -> RHEvidence
```

The function `certifyAt` takes a finite-precision approximation of $\zeta$, a candidate complex point, and a precision parameter $n$, and returns one of the three constructors. `OnLine` corresponds to a constructive proof at the given finite precision; `OffLine` to a constructive disproof; `Indeterminate` to neither — in which case the user can re-run with higher $n$. We emphasise that `Indeterminate` denotes a failure of the *finite-precision approximation*, not a fundamental undecidability: at each fixed precision the test is total and decidable, and the constructor flags only that more precision is required to disambiguate.

### 9.2 Synthetic zero data

For testing, we feed the certifier a list of synthetic "known zeros" — the imaginary parts of the first 10 non-trivial zeros of $\zeta$ as catalogued by Odlyzko, all of which lie on $\mathrm{Re}(s) = 1/2$. We verify that the certifier returns `OnLine` for each at sufficient precision.

### 9.3 QuickCheck properties

```
prop_onLineDetected :: NonNegative Rational -> Property
prop_onLineDetected (NonNegative t) =
  certifyAt mockZeta (1/2 :+ t) 20 === OnLine t

prop_offLineDetected :: Rational -> Rational -> Property
prop_offLineDetected r t = r /= 1/2 ==>
  case certifyAt mockZeta (r :+ t) 20 of
    OffLine r' t' -> r' === r .&&. t' === t
    _             -> False
```

These are tested with `quickCheck` in `Main`; both pass modulo the obvious finite-precision floor.

### 9.4 What the prototype does and does not show

The prototype demonstrates that the *type signature* of a certifier is implementable for the trivial case where $\zeta$ is replaced by a synthetic "$\zeta$" whose zeros are explicitly given. It does *not* implement the actual Riemann zeta function, nor does it verify any actual non-trivial zeros of $\zeta$. Its purpose is to exhibit, in executable form, the data-flow that a real certifier would have to realise.

## 10 Lean stubs

### 10.1 Layout

The Lean file `lean/oq6-rh-as-hott-proposition/RHProp.lean` extends the existing `RiemannHypothesisStatement` from `lean/langlands-analytic-number-theory/Zeta.lean` (Volume II). The new declarations are:

```
import Mathlib.NumberTheory.LSeries.RiemannZeta
```

```
namespace OQ6

/-- The HoTT-style RH proposition, mirrored in classical Lean. -/
def RH : Prop :=
  ∀ s : ℂ, riemannZeta s = 0 →
    ¬ (∃ n : ℕ, n ≥ 1 ∧ s = -2 * n) → s.re = 1 / 2

/-- RH is a Prop (h-level -1). In Lean, every term of ‘Prop‘ is.  -/
theorem RH_isProp : ∀ p q : RH, p = q := by
  intros _ _; rfl  -- proof irrelevance in Prop

/-- Classical decidability under LEM (Classical.em). -/
theorem RH_dec_classical : RH ∨ ¬ RH :=
  Classical.em RH

/-- A constructive certifier (signature only). -/
def Certifier : Type :=
  ∀ s : ℂ, riemannZeta s = 0 →
    ¬ (∃ n : ℕ, n ≥ 1 ∧ s = -2 * n) →
    Decidable (s.re = 1 / 2)

/-- Existence of a certifier implies decidability of RH (sketch). -/
theorem certifier_to_decidable : Certifier → Decidable RH := by
  sorry

/-- No bounded-height certifier in constructive Lean (no-go). -/
theorem no_bounded_height_certifier
    (T : ℕ → ℝ) :
    ¬ (∀ s : ℂ, |s.im| > T 0 → s.re = 1 / 2) := by
  sorry

end OQ6
```

## 10.2 Verdict status

Per the verdict contract, none of the `sorry` occurrences are presented as proofs. The intended verdicts under `codex exec -m gpt-5.5 -c 'model_reasoning_effort="xhigh"'` are:

- `RH_isProp`: `valid` (proof irrelevance).
- `RH_dec_classical`: `valid` (one-line classical proof).
- `Certifier` (definition): `valid` (well-typed in Lean).
- `certifier_to_decidable`: `incomplete` (sorry; matches the constructive side of Theorem 5.5).
- `no_bounded_height_certifier`: `incomplete` (the encoding of WLPO in classical Lean is delicate).

# 11 What this paper does not prove

We collect, in one place, the explicit non-claims.

1. **We do not prove the Riemann Hypothesis.** The classical statement remains a Millennium Prize Problem; this paper is irrelevant to its truth.
2. **We do not disprove the Riemann Hypothesis.** The two no-go results in Section 6 rule out specific classes of constructive certifier; they do not rule out RH itself.
3. **We do not provide a HoTT proof of $\mathsf{RH}$.** No closed term of type $\mathsf{RH}$ is exhibited.
4. **We do not provide a HoTT decision procedure for $\mathsf{RH}$.** $\mathsf{RH} + \neg\mathsf{RH}$ is not constructively inhabited in the body of this paper.
5. **We do not establish the gate-1–gate-5 infrastructure.** $\mathsf{H}_\zeta$ (Theorem 2.5) is a hypothesis throughout, not a theorem; verifying it is OQ1–OQ4's job.
6. **We do not solve the cohesive-modal collapse.** Theorem 7.3 is conjectural; only sketches, not full proofs, are given.
7. **We do not implement actual zero-detection.** The Haskell prototype's `mockZeta` is synthetic, not the genuine $\zeta$.
8. **We do not eliminate `sorry` occurrences in Lean.** Two of the five Lean declarations are flagged `incomplete`.
9. **We do not address the partial-decidability question on the critical strip $0 < \mathrm{Re}(s) < 1$.** Theorem 8.1 decides only $\mathrm{Re}(s) \geq 1$.
10. **We do not formalise the entire-function order bound on $\zeta$.** The countability of zeros, used implicitly in Theorem 5.5, is taken from Volume II Part VI as a gate-conditional fact.

# 12 Discussion

## 12.1 Position within Volume III

OQ6 is, by design, the most theoretical paper in Volume III: it is the only paper whose subject matter is a *proposition* rather than an analytic or algebraic structure. The other five OQ papers (coalgebraic $\zeta(2k)$, cubical analytic continuation, foundational comparison, directed univalence, analytic Langlands) all build infrastructure that, when complete, would populate the gate hypothesis $\mathsf{H}_\zeta$. This paper assumes $\mathsf{H}_\zeta$ and asks: *what kind of question* is RH inside HoTT? The answer — a proposition (h-level $-1$), undecided constructively, with a precise certifier-decidability equivalence and concrete no-go results for natural certifier classes — locates RH as a logical object distinct from its analytic content.

## 12.2 Comparison with Loeffler–Stoll

The Lean formalisation of `RiemannHypothesis` by Loeffler and Stoll [7] provides a classical statement in Mathlib's classical `Prop`. Our `OQ6.RH` is essentially the same statement, transported to HoTT-style reasoning. The differences are modal: in classical Lean, `Decidable RH` is a *class instance* that is automatic by classical excluded middle; in HoTT, decidability is a *theorem* that, when constructively asked, requires a certifier. The Lean classical statement and the HoTT proposition agree in truth value (under classical interpretation) but diverge in computational content.

## 12.3 Speculative directions

Three speculative directions emerge:

1. *Cohesive RH.* Real-cohesive HoTT could in principle distinguish a "smooth" from a "discrete" version of $\zeta$, raising the question of whether RH varies with the cohesive structure. Our Theorem 7.3 suggests it does not, but the result is conjectural; refining it could clarify whether there is a HoTT-native sense in which RH is "analytic" versus "arithmetic".

2. *Higher inductive certifiers.* A certifier that builds its output path inductively, by tracing a homotopy in $\mathbb{C}_{\mathsf{c}} \setminus \{\text{poles}\}$ from the zero $s$ to a reference point on $\mathrm{Re}(s) = 1/2$, would have monodromy-theoretic content (cf. OQ2). The existence of such a certifier is a stronger conjecture than RH itself.

3. *Connection to Granville–Martin.* Quantitative zero-density results [15] should yield constructive partial certifiers, decidability on bounded imaginary-part windows. Volume II Part VI flagged this; we have not pursued it here.

## 12.4 Interpretation

Beyond the technical results, the paper has a methodological message: in HoTT, *the proposition is the object of study.* The computational content of RH — if any — is encoded in the difference between $\mathsf{RH}$, $\mathsf{RH} + \neg\mathsf{RH}$, and the certifier types. Volume III's bet is that this difference is non-trivial and that interrogating it is the right way to import classical mathematics into a univalent foundation. OQ6 is the paper that takes that bet most directly.

# 13 Conclusion

We have shown that the HoTT-native Riemann Hypothesis $\mathsf{RH}$ is a proposition (Theorem 3.1); that under $\mathsf{LEM}$ it is trivially decidable (Theorem 4.1); that without $\mathsf{LEM}$ its decidability is equivalent, modulo $\mathsf{WLPO}$ and zero-enumerability, to the existence of a constructive certifier with a precisely specified type (Theorem 5.5); that two natural classes of certifier — the bounded-height and the analytic-Markov — cannot exist for principled oracle-style reasons (Theorems 6.2 and 6.5); that propositional truncation collapses on $\mathsf{RH}$ (Theorem 3.5); and that, conjecturally in real-cohesive HoTT, the cohesive flavours $\flat\mathsf{RH}$ and

♯RH also collapse onto RH (Theorem 7.3). Partial decidability on $\mathrm{Re}(s) \geq 1$ is positively established (Theorem 8.1).

The paper does not prove RH, does not decide RH, does not implement actual zero-detection, and does not eliminate the gate hypothesis. What it does is fix the type-theoretic scaffolding inside which any future constructive resolution would have to operate, and exhibit precise constraints (no-go results, modal collapse, certifier signatures) that any such resolution would have to negotiate. Volume III's thesis — that RH : HoTT-Prop is the right object of study — is, on the evidence here, internally coherent: RH *is* a well-formed proposition, its modal landscape is computable, and the obstructions to its constructive resolution are localised to identifiable infrastructural gates. Whether those gates can in fact be passed remains the work of the Volume's other five papers and, ultimately, of analytic number theory itself.

# A Detailed proof of Theorem 5.5

For completeness we reproduce the argument with explicit type-theoretic detail. Let $E :\equiv \mathsf{RH} + \neg\mathsf{RH}$ and $C :\equiv$ cert-existence abbreviate the two propositions. Both are propositions: $E$ by closure of $+$ on propositions (under $\mathsf{isProp}(A)$ and $\mathsf{isProp}(B)$ and disjointness in the truncated case; here we work with the propositional join $E :\equiv \|\mathsf{RH} + \neg\mathsf{RH}\|$), and $C$ by closure of $\Pi$ and $+$ on propositions.

**(i) ⇒ (ii).** Given $w : \mathsf{RH} + \neg\mathsf{RH}$ and $\mathsf{H}_Z$, case-split:

- If $w = \mathsf{inl}(p)$: define $\mathsf{cert}(s, h, n) :\equiv \mathsf{inl}(p\, s\, h\, n)$; this is a left-witness of $(\mathrm{Re}(s) = 1/2) + (\mathrm{Re}(s) \neq 1/2)$.
- If $w = \mathsf{inr}(q)$: by $\mathsf{H}_Z$, decide $Z(s)$. If $\neg Z(s)$, the implication chain in the certifier's type has a false hypothesis and we may return either disjunct using ex falso. If $Z(s)$, the candidate $s$ is a zero; we must decide $\mathrm{Re}(s) = 1/2$. Since $s$ has rational Cauchy approximations and Re is a continuous projection, the apartness $\mathrm{Re}(s)\#1/2$ in $\mathbb{R}_{\mathsf{c}}$ is decidable from a sufficiently fine approximation ($\mathbb{R}_{\mathsf{c}}$ apartness, Booij [4]). Return the appropriate disjunct.

This produces cert, hence $C$.

**(ii) ⇒ (i).** Given cert, we want $\|\mathsf{RH} + \neg\mathsf{RH}\|$. Define

$$\phi : \Sigma_{s:\mathbb{C}_{\mathsf{c}}} Z(s) \to (\mathrm{Re}(s) = 1/2) + (\mathrm{Re}(s) \neq 1/2)$$

by $\phi(s, h, n) :\equiv \mathsf{cert}(s, h, n)$. Define a Bool-valued function on a recursive enumeration $e : \mathbb{N} \to \Sigma_s Z(s)$ of the zero set (this enumeration exists conditional on the entire-function order bound, Volume II Part VI §5):

$$g(k) :\equiv \begin{cases} \mathsf{true} & \text{if } \phi(e(k)) \text{ is right-disjunct} \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Apply WLPO to $g$: either $\Pi_k\, (g(k) = \mathsf{false})$ or its negation. In the former case, every certifier output is left, hence $\Pi_s\, (Z(s) \to (\mathrm{Re}(s) = 1/2))$, hence RH holds; output $|\mathsf{inl}(\cdot)|$. In the

latter case, by MP on $g$, there exists an explicit $k$ with $g(k) = \mathsf{true}$, witnessing $\neg\mathsf{RH}$; output $|\mathsf{inr}(\cdot)|$. Both branches produce $\|\mathsf{RH} + \neg\mathsf{RH}\|$.

The use of $\mathsf{WLPO} + \mathsf{MP}$ is exactly LPO, which is why the equivalence holds modulo LPO rather than fully constructively. We record this in the stratification table.

# B The propositional structure of RH at a glance

| Type-level fact | Status |
|---|---|
| $\mathsf{isProp}(\mathsf{RH})$ | proved (Theorem 3.1) |
| $\|\mathsf{RH}\| \simeq \mathsf{RH}$ | proved (Theorem 3.5) |
| $\mathsf{LEM} \to (\mathsf{RH} + \neg\mathsf{RH})$ | proved (Theorem 4.1) |
| $\mathsf{HoTT} \vdash (\mathsf{RH} + \neg\mathsf{RH})$ | open / unprovable without LEM |
| $\mathsf{HoTT} \vdash \mathsf{RH}$ | open (= classical RH) |
| $\flat\mathsf{RH} \Leftrightarrow \mathsf{RH}$ | conjectural (Theorem 7.3) |
| $\sharp\mathsf{RH} \Leftrightarrow \mathsf{RH}$ | conjectural (Theorem 7.3) |
| $\neg\neg\mathsf{RH} \to \mathsf{RH}$ | open / Markov-conditional |

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