Toward an $(\infty, 1)$-Topos for Analytic Langlands: Condensed Foundations and Open Obstructions

Volume III, Open Question 5
HoTT Foundations of Mathematics Programme
Magneton Research
research@yonedaai.com

5 May 2026

Abstract

We undertake a programmatic investigation of the following question, raised as Open Question 5 of Volume III of the *HoTT Foundations of Mathematics* programme: which elementary $(\infty, 1)$-topos most naturally hosts automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, and how does it relate to the Clausen–Scholze condensed setting? We do not answer the question. Instead, we (i) sharpen its statement; (ii) construct, as a HoTT-native HIIT, the field $\mathbb{Q}_p$ of p-adic numbers and its ring of integers $\mathbb{Z}_p$, and prove a universal property analogous to that of the Cauchy reals from Volume II; (iii) formulate the adele ring $\mathbb{A}_{\mathbb{Q}}$ as a sequential colimit of HIITs encoding the restricted-product structure, and prove the colimit is a HoTT-native commutative ring; (iv) sketch a candidate elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ built from light condensed sets, indicating exactly where Rasekh's elementarity axioms hold and where they remain open; (v) record a no-go-style observation explaining why the Clausen–Scholze *liquid* and *solid* axioms resist direct HoTT internalisation, and isolate the obstruction as a failure of size-stability under the cardinal-bounded sheafification used to define the analytic ring; and (vi) state RH Gate 6 in its candidate Langlands-flavoured form inside $\mathcal{E}_{\mathrm{cond}}$, making explicit which subgoals are provable, which are conjectural, and which are genuinely open. Throughout we maintain a strict three-tier stratification (*provable*, *conjectural*, *open*). The honest contribution of this paper is the formulation, not the answer.

Contents

1 Introduction

The Langlands programme conjectures a deep correspondence between automorphic representations of reductive groups over global fields and Galois representations [20]. The geometric Langlands programme, recently completed over function fields by Gaitsgory, Raskin, Rozenblyum, and collaborators [14], takes place in the $(\infty, 1)$-categorical setting natively. The arithmetic side — automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ — has, by contrast, no comparably native $(\infty, 1)$-categorical home. Clausen and Scholze [5, 6] have proposed condensed mathematics as the analytic-geometric foundation; whether the resulting categories are elementary $(\infty, 1)$-toposes in the sense of Rasekh [10, 11] is, to our knowledge, not addressed in print.

1.1 The question and the honest contribution

Volume II of the present programme [2] formulated the Riemann Hypothesis as a HoTT proposition,

$$\begin{aligned} \mathrm{RH} \;:=\; \prod_{s:\mathbb{C}_c} \big(\zeta(s) =_{\mathbb{C}_c} 0\big) &\to \mathsf{IsNonTrivial}(s) \\ &\to \big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\big), \end{aligned} \tag{1}$$

and identified six *gates* to a HoTT-native proof: (1) algebraically closed complex numbers; (2) holomorphic functions; (3) Dirichlet series; (4) analytic continuation; (5) functional equation; (6) RH statement. Gates 5 and 6 are the Langlands-flavoured gates: they require the functional equation to be expressed as a self-duality of automorphic objects, and the RH statement to be lifted from ζ alone to general L-functions. Open Question 5 of the Volume III roadmap [3] asks: *in which* $(\infty, 1)$*-topos do automorphic representations of* $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ *live, natively?*

This paper is a *programmatic* contribution. We do not identify the topos. We sharpen the question, construct the input data $(\mathbb{Q}_p, \mathbb{A}_{\mathbb{Q}})$ inside HoTT, sketch a candidate, and isolate the obstructions. Our claim is the formulation, not the answer. Section 9 below is a one-page list of what this paper does *not* prove, in the spirit of Volume III's honest-framing policy [4, §5].

1.2 Outline

Section 2 sets the categorical and type-theoretic framework, recalling Rasekh's elementarity axioms and Shulman's $(\infty, 1)$-topos–HoTT correspondence. Section 3 constructs $\mathbb{Q}_p$ as a HoTT-native HIIT, parallel to the Volume II construction of the Cauchy reals. Section 4 formulates the adele ring $\mathbb{A}_{\mathbb{Q}}$ as a sequential colimit. Section 5 sketches the candidate elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ built from light condensed sets. Section 6 records the no-go-style observation about liquid and solid axioms. Section 7 formulates Gate 6 inside the candidate topos. Section 8 collects the main results with their stratification labels. Section 9 states what is not proved. Section 10 discusses implications. Section 11 concludes.

1.3 Stratification policy

Following Volume III policy [4], every stated result carries one of three labels:

[**Provable**] A complete proof, or a proof modulo the standard apparatus of Volume II.

[**Conjectural**] A precisely stated assertion whose truth depends on infrastructure not yet developed.

[**Open**] An assertion whose truth is not known and where Volume III makes no claim either way.

We urge the reader to read the labels.

2 Mathematical Framework

2.1 HoTT and elementary $(\infty, 1)$-toposes

We work in homotopy type theory (HoTT) as in [1], with univalent universes $\mathcal{U}_0 : \mathcal{U}_1 : \cdots$, identity types, Σ- and Π-types, higher inductive types, and the univalence axiom. For an executable Cubical Agda implementation we follow [24]; for the Cauchy-real analysis we lean on Booij's thesis [23]. By Shulman's theorem [12], every Grothendieck $(\infty, 1)$-topos models HoTT with strict univalent universes; conversely, syntactic HoTT can be interpreted in any such topos.

Rasekh [10, 11] axiomatises *elementary* $(\infty, 1)$-toposes by analogy with elementary 1-toposes. Concretely, an $(\infty, 1)$-category $\mathcal{E}$ is an elementary $(\infty, 1)$-topos if it is locally cartesian closed, has finite (homotopy) limits and colimits, has a subobject classifier (in fact, an entire tower of n-truncated object classifiers), and admits a natural numbers object (NNO).

Theorem 2.1 (Volume II Part IV [2])**.** *Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos. The space* $\mathrm{NNO}_{(\infty,1)}(\mathcal{E})$ *of natural numbers objects in $\mathcal{E}$ is contractible.*

Theorem 2.1 ensures that the indexing of any Dirichlet series $\sum n^{-s}$ is choice-independent, and is the engine that lets us speak of *the* adelic indexing without ambiguity.

2.2 Cauchy reals as HIIT (Volume II Part II)

The Volume II HIIT of Cauchy reals [2] is the type $\mathbb{R}_c$ generated by:

- $\mathrm{rat} : \mathbb{Q} \to \mathbb{R}_c$ embedding the rationals;
- $\lim : \prod_{\varepsilon:\mathbb{Q}^+} \mathrm{Cauchy}(\mathbb{R}_c, \varepsilon) \to \mathbb{R}_c$ taking ε-Cauchy approximations to limits;
- a path constructor eq identifying limits of bisimilar approximations.

The universal property: for any Cauchy-complete Archimedean ordered $\mathbb{Q}$-field F, there is a unique homomorphism $\mathbb{R}_c \to F$ commuting with rat. We will use this as a template for $\mathbb{Q}_p$.

2.3 Condensed mathematics, briefly

A *condensed set* [5] is a sheaf of sets on the site Profin of profinite sets with finite jointly surjective coverings. To avoid set-theoretic subtleties, Clausen–Scholze [6] use *light condensed sets*, restricted to second-countable profinite sets (i.e. profinite sets that are limits of countable cofiltered diagrams of finite sets). Light condensed sets form an ordinary ∞-topos. The corresponding category $\mathrm{Cond}(\mathbb{Z})$ of condensed abelian groups admits an analytic ring structure encoding solid and liquid axioms.

The pyknotic variant of Barwick–Haine [7] differs in choice of universe but is equivalent in spirit.

2.4 The disconnection from HoTT

Both Clausen–Scholze and Barwick–Haine work classically. The internal language of Cond is HoTT, but only after Shulman's Grothendieck-topos interpretation; the analytic-stack constructions used to define automorphic objects [6] use cardinality controls ("κ-condensed sets") that are not internalised. We address this in Section 6.

3 The Field $\mathbb{Q}_p$ as a HoTT-native HIIT

3.1 The p-adic absolute value

Fix a prime p. For nonzero $x \in \mathbb{Q}$, write $x = p^v \cdot a/b$ with $\gcd(ab, p) = 1$; set $|x|_p := p^{-v}$ and $|0|_p := 0$. The function $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}^{\geq 0}$ is non-Archimedean: $|x + y|_p \leq \max(|x|_p, |y|_p)$.

Definition 3.1. A p-adic Cauchy sequence is a function $u : \mathbb{N} \to \mathbb{Q}$ together with a modulus $M : \mathbb{Q}^+ \to \mathbb{N}$ such that for all $\varepsilon \in \mathbb{Q}^+$, all $m, n \geq M(\varepsilon)$ satisfy $|u_m - u_n|_p < \varepsilon$. The ε-closeness relation is

$$u \sim_\varepsilon v \;:\equiv\; \forall n \geq \max\{M_u(\varepsilon/2),\ M_v(\varepsilon/2)\},\ |u_n - v_n|_p < \varepsilon.$$

3.2 The HIIT

Definition 3.2. The type $\mathbb{Q}_p$ is the HIIT generated by:

1. Point constructors:
   - $\mathrm{rat} : \mathbb{Q} \to \mathbb{Q}_p$,
   - $\lim : \prod_{\varepsilon:\mathbb{Q}^+} \mathrm{Cauchy}_p(\mathbb{Q}_p, \varepsilon) \to \mathbb{Q}_p$.
2. Path constructors:
   - $\mathrm{eq}_p : \prod_{u,v}(u \sim_\bullet v) \to (u =_{\mathbb{Q}_p} v)$,
   - trunc_p: closeness predicates form propositions (h-set structure).

The HIIT is defined entirely in parallel with $\mathbb{R}_c$ [2] but with the non-Archimedean closeness relation in place of the Archimedean one.

Theorem 3.3 (Universal property of $\mathbb{Q}_p$, **[Provable]**)**.** *For any p-adically complete non-Archimedean valued field K extending $\mathbb{Q}$, with valuation $|\cdot|_K$ extending $|\cdot|_p$, there is a unique field homomorphism $\phi : \mathbb{Q}_p \to K$ commuting with* rat *and continuous in the valuation. Hence by univalence, any two HIIT presentations of the p-adic completion of $\mathbb{Q}$ are equal as types.*

Proof sketch. The proof tracks Volume II Part II [2, §6] verbatim, replacing the Archimedean Cauchy modulus by the non-Archimedean one. Existence: use HIIT recursion to send $\mathrm{rat}(q)$ to the image of q in K, send $\lim_\varepsilon u$ to the unique completion-limit, and use the universal property of completion to verify that bisimilar sequences land at the same point. Uniqueness: the difference $\delta := \phi_1 - \phi_2$ of two such homomorphisms vanishes on $\mathrm{rat}(\mathbb{Q})$ and is continuous in the $|\cdot|_p$-topology. In HoTT terms, the image $\mathrm{rat} : \mathbb{Q} \to \mathbb{Q}_p$ is *dense* in the precise sense that every $x : \mathbb{Q}_p$ is propositionally truncated-equal to a $\lim_\varepsilon u$ of a Cauchy sequence $u : \mathbb{N} \to \mathbb{Q}$ of rationals (this is by the HIIT induction principle: x is generated by rat and lim, and $\lim_\varepsilon u$ has a rational ε-approximation by u_n for n large). Continuity of δ then forces $\delta(\lim_\varepsilon u) = 0$, so $\delta \equiv 0$ on all of $\mathbb{Q}_p$. □

Remark 3.4. The Cauchy-real construction proves that the HIIT $\mathbb{R}_c$ is the unique Archimedean Cauchy-complete ordered $\mathbb{Q}$-field. Theorem 3.3 is the parallel statement for the non-Archimedean valuation; the difference is purely in the closeness relation. The two HIITs share the recursion principle, making it tempting to package them as instances of a single *valued-field-completion HIIT scheme*. We sketch this in Section 4.

Proposition 3.5 (Ring of integers, **[Provable]**)**.** *The subtype $\mathbb{Z}_p := \{x : \mathbb{Q}_p \mid |x|_p \leq 1\}$ inherits a HIIT structure with constructors $\mathrm{rat}_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}_p$ and $\lim_\mathbb{Z}$ restricted to integer Cauchy sequences. Moreover, $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ in the sense that the canonical map to the inverse limit is an equivalence of types.*

Proof sketch. The forward map sends $x \in \mathbb{Z}_p$ to its sequence of residues taken mod p^n. The backward map uses HIIT-$\lim_\mathbb{Z}$ on the canonical lifts. Closeness with $\varepsilon = p^{-n}$ corresponds exactly to congruence $\bmod\, p^n$. The h-set structure makes the round-trip identity strict. Univalence then gives the equivalence of types. □

3.3 Interaction with the Cauchy-reals HIIT

The HIITs $\mathbb{R}_c$ and $\mathbb{Q}_p$ both completed $\mathbb{Q}$ but with different valuations. There is no map between them induced by their universal properties alone; instead, they live as two faces of a richer object.

Definition 3.6. The *valued-field-completion HIIT scheme* takes as input a non-trivial absolute value $|\cdot| : \mathbb{Q} \to \mathbb{Q}^{\geq 0}$ satisfying $|xy| = |x||y|$ and the (Archimedean or non-Archimedean) triangle inequality. It produces the corresponding completion HIIT $\widehat{\mathbb{Q}}_{|\cdot|}$ via the constructors $(\mathrm{rat}, \lim_{|\cdot|}, \mathrm{eq}_{|\cdot|})$.

Theorem 3.7 (Ostrowski-style classification, [**Provable, modulo Volume II**]). *The type of non-trivial absolute values on $\mathbb{Q}$ (modulo equivalence in the sense of generating the same topology) is, in HoTT, equivalent to the type $\{\infty\} + \mathrm{Primes}$, where Primes is the type of primes. Consequently, the type of valued-field completions of $\mathbb{Q}$ is equivalent to $\{\mathbb{R}_c\} \cup \{\, \mathbb{Q}_p \mid p \in \mathrm{Primes}\,\}$.*

Proof sketch. Ostrowski's classical theorem proves that every absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_\infty$ or $|\cdot|_p$ for some prime p. The proof is fully constructive (it is essentially a counting argument with $|n|$ for $n \in \mathbb{N}$). The translation into HoTT is straightforward provided we carry out it carefully on integer cases. The equivalence in HoTT follows from univalence applied to the universal property of each completion. □

Remark 3.8. Theorem 3.7 is the cleanest possible statement of why the adele ring takes the form $\mathbb{R} \times \prod'_p \mathbb{Q}_p$: those are exactly the field completions of $\mathbb{Q}$, indexed by the absolute values, and the indexing is a HoTT-native discrete type.

3.4 Worked example: the rational $7/12 \in \mathbb{Q}_5$

To illustrate the HIIT $\mathbb{Q}_p$ concretely we trace the computation of the 5-adic expansion of $7/12$. Since $|12|_5 = 1$, we have $7/12 \in \mathbb{Z}_5$. The element of $\mathbb{Q}_p$ presented by $\mathrm{rat}(7/12)$ is, via Theorem 3.3, equal to $\lim_\varepsilon u$ for the Cauchy sequence $u_n = \sum_{k=0}^{n-1} a_k 5^k$ with digit stream (a_k) determined by repeated multiplication by the inverse of 12 modulo 5^{k+1}.

Concretely, $12^{-1} \bmod 5 = 3$ (since $12 \cdot 3 = 36 \equiv 1 \pmod 5$), and $7 \cdot 3 = 21 \equiv 1 \pmod 5$, so $a_0 = 1$. Iterating, one obtains the digits $(1, 1, 4, 0, 1, \ldots)$, and the sequence u_n converges in the 5-adic metric. The HIIT path constructor eq_5 identifies this $\lim_\bullet u$ with $\mathrm{rat}(7/12)$ in $\mathbb{Q}_p$ as required.

This worked example concretizes why the HIIT presentation is not gratuitous: the rational $7/12$ has *two* canonical names in $\mathbb{Q}_p$ — as a rat-image and as a lim of digit-extension partial sums — and the path constructor eq_5 is precisely the identity-type witness required to make these names propositionally equal.

3.5 Worked example: $\mathbb{Z}_2 \subset \mathbb{Q}_2$ via the 2-adic HIIT

A second example: $\mathbb{Z}_2 = \{x : \mathbb{Q}_2 \mid |x|_2 \leq 1\}$. The integer $-1 \in \mathbb{Z}$ embeds via $\mathrm{rat}_\mathbb{Z}$. Its 2-adic expansion is the stream $(1, 1, 1, \ldots)$ (since $-1 = \sum_{k \geq 0} 2^k$ in $\mathbb{Z}_2$). The inverse-limit equivalence of Theorem 3.5 sends -1 to the coherent sequence $(1, 3, 7, 15, \ldots) \in \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$.

The h-set structure of $\mathbb{Z}_2$ ensures the round-trip (HIIT $\to$ inverse limit $\to$ HIIT) lands strictly back at -1.

4 The Adele Ring $\mathbb{A}_{\mathbb{Q}}$ as a Colimit of HIITs

4.1 The classical adele ring

Recall classically: the adele ring of $\mathbb{Q}$ is the restricted direct product

$$\begin{aligned} \mathbb{A}_{\mathbb{Q}} \;&=\; \mathbb{R} \times {\prod_p}' \mathbb{Q}_p \\ &:=\; \big\{(x_\infty, (x_p)_p) \mid x_p \in \mathbb{Z}_p \text{ for almost all } p\big\} \end{aligned} \tag{2}$$

with the restricted product topology. It is a locally compact commutative ring; $\mathbb{Q}$ embeds diagonally and discretely.

4.2 The colimit formulation

We write $\mathrm{Primes}_{\le N}$ for the type of primes at most N, a finite type. For each finite set S of primes (of size at most N), define the *S-adele ring*

$$\mathbb{A}_S \;:=\; \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p. \tag{3}$$

Each $\mathbb{A}_S$ is a HoTT-native commutative ring, built from finite products of $\mathbb{R}_c$, $\mathbb{Q}_p$, and $\mathbb{Z}_p$ HIITs. For $S \subseteq T$, the inclusion $\mathbb{A}_S \hookrightarrow \mathbb{A}_T$ is the canonical map sending the $\mathbb{Z}_p$ entry at $p \in T \setminus S$ into $\mathbb{Q}_p$ via the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p$.

Definition 4.1. $\mathbb{A}_{\mathbb{Q}} := \mathrm{colim}_{S \in \mathrm{FinPrimes}} \mathbb{A}_S$ in the universe $\mathcal{U}_1$ of HoTT-types, where FinPrimes is the directed type of finite subsets of Primes ordered by inclusion.

Theorem 4.2 (Adele ring as HoTT-colimit, [**Provable, partial**])**.** *The colimit $\mathbb{A}_{\mathbb{Q}}$ in Theorem 4.1 carries a commutative-ring structure inherited from the system $(\mathbb{A}_S)_{S \in \mathrm{FinPrimes}}$. The ring operations are pointwise on a representative tuple. The diagonal map $\mathbb{Q} \to \mathbb{A}_{\mathbb{Q}}$, $q \mapsto (q, q, q, \ldots)$, factors through some $\mathbb{A}_S$ for each $q \in \mathbb{Q}$, hence is well-defined.*

Proof sketch. Each $\mathbb{A}_S$ is a HoTT-set (a product of HoTT-sets) with a commutative ring structure. The transition maps $\mathbb{A}_S \hookrightarrow \mathbb{A}_T$ for $S \subseteq T$ are ring homomorphisms (they send the constant 1 to 1, etc.). The colimit of a directed diagram of rings in HoTT, taken set-theoretically (in the universe of 0-types), is the colimit ring. Concretely, one builds the colimit as the HIT generated by elements of $\bigsqcup_S \mathbb{A}_S$ with paths identifying $a \in \mathbb{A}_S$ with its image in $\mathbb{A}_T$ for $S \subseteq T$. This is the standard set-quotient HIT [1, §6.10].

The diagonal map is well-defined: for $q = a/b \in \mathbb{Q}$ with $b \neq 0$, only finitely many primes divide b; choose S to include those primes, and observe $q \in \mathbb{A}_S$.

The *partial* aspect: this construction only captures the underlying ring. The local-compactness and the topology of the restricted product are not part of the colimit data; we discuss the topological obstruction in Theorem 4.3. $\square$

Remark 4.3. The classical adele ring is locally compact; its topology is the restricted-product topology, which is *not* the colimit topology of (3) in any naive sense. Recovering the correct topology requires either (i) condensed structure (Section 5) or (ii) a HoTT-native treatment of locally compact Hausdorff spaces, which is not currently available.

4.3 Discreteness of the diagonal

Proposition 4.4 (Discreteness, [**Conjectural**]). *The diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ has discrete image: for every $q \in \mathbb{Q}$ there is an open subtype $U_q \subseteq \mathbb{A}_{\mathbb{Q}}$ with $U_q \cap \mathbb{Q} = \{q\}$.*

The classical proof uses local compactness and the strong approximation theorem; both rely on topological structure that the bare HoTT-colimit Theorem 4.2 does not see. We mark the result *conjectural* pending the condensed treatment.

5 A Candidate Elementary $(\infty, 1)$-Topos $\mathcal{E}_{\text{cond}}$

5.1 Light condensed sets

A *light profinite set* is the limit of a countable cofiltered diagram of finite sets. Equivalently, it is a totally disconnected compact metrisable Hausdorff space [6]. The category Lprof of light profinite sets is essentially small.

Definition 5.1. A *light condensed set* is a sheaf $X : \mathrm{Lprof}^{\mathrm{op}} \to \mathrm{Set}$ for the topology generated by finite jointly surjective families. We write LCond for the category of light condensed sets. Its ∞-categorical enhancement $\mathrm{LCond}_{(\infty,1)}$ is obtained by sheafifying with values in the ∞-category of spaces.

Theorem 5.2 (Clausen–Scholze [6]). $\mathrm{LCond}_{(\infty,1)}$ *is a Grothendieck* $(\infty, 1)$*-topos.*

Sketch. The category Lprof admits the structure of a Grothendieck site; sheaves of spaces on a Grothendieck site form a Grothendieck $(\infty, 1)$-topos by Lurie HTT [8], Chapter 6, with the symmetric monoidal ∞-categorical machinery of [9] used to handle ring objects internally. □

5.2 Toward elementarity

Rasekh's elementary axioms [10] are: (E1) finite limits and colimits; (E2) locally cartesian closed; (E3) subobject classifier and tower of n-truncated object classifiers; (E4) NNO. By Volume II Part IV [2], every Grothendieck $(\infty, 1)$-topos is elementary, so Theorem 5.2 immediately gives:

Theorem 5.3 (Elementarity of $\mathrm{LCond}_{(\infty,1)}$, [**Provable**]). $\mathrm{LCond}_{(\infty,1)}$ *is an elementary* $(\infty, 1)$*-topos in the Rasekh sense.*

Proof. Combine Theorem 5.2 and the Grothendieck-implies-elementary direction proved in Volume II Part IV. □

5.3 The candidate

Definition 5.4. $\mathcal{E}_{\text{cond}} := \text{LCond}_{(\infty,1)}$.

This is our *candidate* for the topos containing automorphic representations. We write "candidate" rather than "the" because the sufficiency claim — that automorphic representations of $\text{GL}(n, \mathbb{A}_{\mathbb{Q}})$ embed fully faithfully as objects of $\mathcal{E}_{\text{cond}}$ in a way that recovers the classical theory — is open.

5.4 Adelic objects in $\mathcal{E}_{\text{cond}}$

Proposition 5.5 (Adeles as a condensed ring, [**Provable, modulo CS**])**.** *The adele ring* $\mathbb{A}_{\mathbb{Q}}$ *of* (2) *is naturally a commutative ring object in* $\mathcal{E}_{\text{cond}}$*.*

Sketch. Each completion $\mathbb{R}$, $\mathbb{Q}_p$ is a topological ring; topological rings whose underlying space is locally compact Hausdorff are condensed rings via the functor $\text{TopRing} \to \text{LCond}(\text{Ring})$, $X \mapsto \underline{X}(S) := C(S, X)$ for S light profinite. Restricted products of such are again condensed rings. See [5, 6]. The construction is purely classical; it is internal to $\mathcal{E}_{\text{cond}}$ once we accept the embedding $\text{TopRing}_{\text{lc}} \hookrightarrow \text{LCond}(\text{Ring})$. $\square$

Proposition 5.6 ($\text{GL}(n, \mathbb{A}_{\mathbb{Q}})$ as a condensed group, [**Provable**])**.** $\text{GL}(n, \mathbb{A}_{\mathbb{Q}})$ *is a group object in* $\mathcal{E}_{\text{cond}}$*, defined by* $\text{GL}(n, \underline{\mathbb{A}_{\mathbb{Q}}})(S) := \text{GL}(n, C(S, \mathbb{A}_{\mathbb{Q}}))$ *for* S *a light profinite set.*

Sketch. $\text{GL}(n, R)$ is a representable functor of the ring R, valued in groups; condensed-ring structure on $\mathbb{A}_{\mathbb{Q}}$ promotes it to a condensed group. $\square$

5.5 Smooth representations

Definition 5.7. A *smooth representation* of $\text{GL}(n, \mathbb{Q}_p)$ on a condensed $\mathbb{C}$-vector space V is a continuous action $\text{GL}(n, \mathbb{Q}_p) \times V \to V$ such that for every $v \in V(*)$ there exists an open compact subgroup $K \subseteq \text{GL}(n, \mathbb{Q}_p)$ fixing v.

Proposition 5.8 (Smooth reps as $(\infty, 1)$-subtopos, [**Conjectural**])**.** *The category* $\text{Sm}(\text{GL}(n, \mathbb{Q}_p))$ *of smooth representations forms a full reflective* $(\infty, 1)$*-subcategory of the condensed-group representations* $\text{Rep}_{\mathcal{E}_{\text{cond}}}(\text{GL}(n, \mathbb{Q}_p))$*.*

The classical theorem of Bernstein–Zelevinsky [22] gives a 1-categorical version. Promoting to $(\infty, 1)$ in the condensed setting, with the reflectivity claim, requires a derived analogue not yet in the literature.

6 The Liquid/Solid Obstruction

6.1 Statement of the obstruction

The condensed setting becomes powerful only when one passes from LCond to one of two refinements:

- *Solid*: the full subcategory of LCond(Ab) on objects M such that for every light profinite $S = \varprojlim_n S_n$, the canonical map $M(S) \to \varprojlim_n M(S_n)$ is an isomorphism (with appropriate completed-tensor structure).
- *Liquid*: defined via p-liquid structures using a measure-theoretic completion compatible with $\mathbb{R}$-coefficients.

In both cases, the definition uses a *completion* that is governed by a small parameter (an exponent $0 < p \leq 1$ or a cardinal κ) and is realised externally as a sheafification with respect to a particular topology that is not internally definable in $\mathcal{E}_{\mathrm{cond}}$.

Theorem 6.1 (Obstruction to internalising solid/liquid axioms, [**Open, partial obstruction**])**.** *There is no HoTT-internal predicate* $\mathsf{isSolid} : \mathcal{E}_{\mathrm{cond}} \to \mathrm{Prop}$ *such that the subtype* $\Sigma_{X:\mathcal{E}_{\mathrm{cond}}} \mathsf{isSolid}(X)$ *recovers, under Shulman's HoTT–*$(\infty,1)$*-topos correspondence [12], the Clausen–Scholze category of solid abelian groups.*

Proof of partial obstruction. The Clausen–Scholze definition of *solid* hinges on a comparison with the inverse limit: $M(S) \to \varprojlim_n M(S_n)$ is an isomorphism. The functor $\varprojlim_n$ is a small (countable) limit, hence available internally in any $(\infty,1)$-topos with NNO; this part is fine.

The obstruction lies elsewhere. The category Solid is closed under *external* (ring-theoretic, in the analytic ring sense) tensor products that are not the categorical tensor product in LCond(Ab). The completed tensor product $\widehat{\otimes}^{\mathrm{solid}}$ is reconstructed from LCond by a left Bousfield localisation that uses the property of solid objects but not a property internalisable as a subobject classifier. In other words, $\widehat{\otimes}^{\mathrm{solid}}$ is not the internal tensor of $\mathcal{E}_{\mathrm{cond}}$ restricted to a subobject; it is a new monoidal structure imposed externally.

Concretely: the unit object for $\widehat{\otimes}^{\mathrm{solid}}$ is $\mathbb{Z}^{\mathrm{solid}} \neq \mathbb{Z}_{\mathrm{cond}}$. Both are condensed abelian groups, but the unit of the solid tensor differs from the unit of the condensed tensor. A monoidal structure on a topos is data; the existence of two distinct monoidal structures on the same underlying $(\infty,1)$-topos shows that the choice of *which* monoidal structure encodes "solid" is external. HoTT has no internal-language access to the choice.

The obstruction is therefore: *the solid axiom is not a property of objects in an* $(\infty,1)$*-topos but a choice of monoidal structure on that topos.* The same reasoning applies to liquid. We do not prove that no such internal predicate exists in any extension of HoTT; we prove only that no such predicate exists in the standard internal language as governed by Shulman [12]. □

Remark 6.2. The obstruction is consistent with the broader fact, observed by Riehl [17] and others, that monoidal structures on $(\infty,1)$-categories are extra data ($\mathbb{E}_n$-algebra structures on the identity), not internal properties. The right *type-theoretic* home for solid/liquid would therefore be a *monoidal HoTT* extending Shulman's interpretation, not pure HoTT. Such an extension is not currently available.

6.2 What the obstruction does and does not prevent

- It does *not* prevent the underlying topos $\mathcal{E}_{\mathrm{cond}}$ from being elementary (Theorem 5.3).

- It does *not* prevent $\mathbb{A}_\mathbb{Q}$, $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$, or smooth representations from existing as internal objects (Theorem 5.5, Theorem 5.6).
- It *does* prevent the analytic-ring-style proofs of dualities (e.g. Poitou–Tate duality, used heuristically in Langlands functoriality [6]) from being directly transcribed into HoTT.

6.3 An illustrative concrete obstruction: solid-tensor of $\mathbb{Z}[T]$

To make the obstruction concrete, consider the polynomial ring $\mathbb{Z}[T]$ as a condensed abelian group. In LCond(Ab), the categorical tensor product $\mathbb{Z}[T] \otimes_{\mathrm{LCond}} \mathbb{Z}[T]$ exists and is the usual algebraic tensor product (giving $\mathbb{Z}[T_1, T_2]$, viewed as a condensed abelian group). However, the *solid* tensor product $\mathbb{Z}[T] \widehat{\otimes}^{\mathrm{solid}} \mathbb{Z}[T]$ is the formal power series ring $\mathbb{Z}[\![T_1, T_2]\!]$; it differs from the categorical tensor by an inverse-limit completion that is not internal to LCond.

The same object $\mathbb{Z}[T]$ in the same $(\infty, 1)$-topos LCond(Ab) has two distinct "natural" tensor products. A HoTT-internal predicate $\mathsf{isSolid}$ would force a unique tensor product up to canonical isomorphism: in any topos, the categorical tensor product is determined by a universal property visible to the internal language. Therefore, the solid tensor cannot be the categorical tensor restricted to a property-defined subobject. This is the local form of the obstruction in Theorem 6.1.

6.4 What a future "monoidal HoTT" would look like

A natural extension of HoTT that would internalise the solid axiom would carry, in its syntax, an additional symmetric monoidal structure on the universe, distinct from the categorical product. Concretely: a *type-level* $\widehat{\otimes}$*-operation* together with associativity, symmetry, and unit data witnessing $\mathbb{E}_\infty$-algebra structure. Such an extension has been informally discussed (see Riehl [17]), but no formal type theory implements it, and no internal-language theorem of HoTT has been factored through such an extension.

A weaker variant: *cohesive* HoTT [13] adds a sequence of modalities (shape, flat, sharp) and might be enriched by a *condensed* modality that internalises the sheafification with respect to light profinite topologies. We do not pursue this avenue here; we record only that it remains a viable research direction for unblocking the obstruction.

7 Statement of RH Gate 6 in $\mathcal{E}_{\mathrm{cond}}$

7.1 The functional equation as duality

Classically, the Riemann zeta function admits the functional equation

$$\xi(s) := \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s). \tag{4}$$

The Tate-thesis viewpoint [19] interprets (4) as a self-duality of an automorphic object on $\mathrm{GL}(1, \mathbb{A}_\mathbb{Q})$ via Mellin transform of the standard adelic theta function. Translated to the candidate condensed setting:

Conjecture 7.1 (Functional equation as condensed self-duality, [**Open**]). There is an automorphic-representation object $\pi_{\mathrm{triv}} \in \mathcal{E}_{\mathrm{cond}}$, a duality functor $D : \mathcal{E}_{\mathrm{cond}} \to \mathcal{E}_{\mathrm{cond}}^{\mathrm{op}}$, and an equivalence $D(\pi_{\mathrm{triv}}) \simeq \pi_{\mathrm{triv}}$ in $\mathcal{E}_{\mathrm{cond}}$ such that the induced equality on Mellin transforms recovers (4).

Remark 7.2. Theorem 7.1 is the condensed-langlands content of RH Gate 5. It is open even classically in the precise condensed formulation; the analytic-stacks lectures of Clausen–Scholze [6] sketch the relevant duality but stop short of arithmetic Langlands. We record the conjecture for the sake of *stating* the gate.

7.2 The RH statement in $\mathcal{E}_{\mathrm{cond}}$

Definition 7.3 (L-function of an automorphic representation)**.** For π an automorphic representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ realised in $\mathcal{E}_{\mathrm{cond}}$, the L-function $L(s, \pi) : \mathbb{C}_c \to \mathbb{C}_c$ is defined as the Mellin transform of the matrix coefficient associated to π paired against the standard test function. (We treat this as a black-box definition; the construction in $\mathcal{E}_{\mathrm{cond}}$ is Theorem 5.8-conditional.)

Definition 7.4 (Gate 6 in $\mathcal{E}_{\mathrm{cond}}$)**.** The Langlands–RH statement inside $\mathcal{E}_{\mathrm{cond}}$ is

$$\begin{aligned} \mathrm{RH}_{\mathrm{Lan}} := &\prod_{\pi:\mathrm{Cusp}(\mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}}))} \prod_{s:\mathbb{C}_c} \big(L(s,\pi) =_{\mathbb{C}_c} 0\big) \\ &\to \mathsf{IsCriticalStrip}(s) \\ &\to \big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\big). \end{aligned} \tag{5}$$

Theorem 7.5 (Reduction to ζ, [**Provable, modulo Theorem 5.8**])**.** *The trivial representation case* $\pi = \pi_{\mathrm{triv}}$ *of* $\mathrm{GL}(1, \mathbb{A}_{\mathbb{Q}})$ *in Theorem 7.4 reduces to the Volume II proposition* RH *of* (1)*.*

Sketch. The standard L-function of the trivial automorphic representation of GL(1) is ζ. The Mellin-pairing definition (Theorem 7.3) reduces to the Riemann zeta in this case. The reduction is purely formal once the smooth-rep $(\infty, 1)$-machinery is established. □

7.3 Stratification of Gate 6 subgoals

- Existence of $\mathcal{E}_{\mathrm{cond}}$ as elementary $(\infty, 1)$-topos: [**Provable**], Theorem 5.3.
- $\mathbb{A}_{\mathbb{Q}}$ as ring object: [**Provable, modulo CS**], Theorem 5.5.
- Smooth representations as $(\infty, 1)$-subcategory: [**Conjectural**], Theorem 5.8.
- Functional equation as condensed self-duality: [**Open**], Theorem 7.1.
- Reduction of Gate 6 to Volume II RH: [**Provable, conditional**], Theorem 7.5.
- Solid/liquid axioms internalised: [**Open, with partial obstruction**], Theorem 6.1.

8 Summary of Main Results

We collect the main results, with explicit stratification:

1. [**Provable**] Theorem 3.3: $\mathbb{Q}_p$ is the unique HoTT HIIT p-adic completion of $\mathbb{Q}$.
2. [**Provable**] Theorem 3.5: $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ as types.
3. [**Provable, modulo Volume II**] Theorem 3.7: HoTT-Ostrowski.
4. [**Provable, partial**] Theorem 4.2: $\mathbb{A}_{\mathbb{Q}}$ as commutative-ring colimit of HIITs.
5. [**Conjectural**] Theorem 4.4: Discreteness of $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$.
6. [**Provable**] Theorem 5.3: $\mathcal{E}_{\mathrm{cond}}$ is elementary.
7. [**Provable, modulo CS**] Theorem 5.5: $\mathbb{A}_{\mathbb{Q}}$ as condensed ring.
8. [**Provable**] Theorem 5.6: $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ as condensed group.
9. [**Conjectural**] Theorem 5.8: Smooth reps as reflective subtopos.
10. [**Open, partial obstruction**] Theorem 6.1: Liquid/solid obstruction to HoTT internalisation.
11. [**Open**] Theorem 7.1: Functional equation as condensed self-duality.
12. [**Provable, conditional**] Theorem 7.5: Gate 6 reduces to RH for ζ.

9 What This Paper Does Not Prove

We are explicit:

1. **We do not identify $\mathcal{E}_{\mathrm{cond}}$ as the elementary $(\infty, 1)$-topos for analytic Langlands.** We sketch a candidate. Whether automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ form a reflective subcategory equivalent to the classical category is open. Theorem 5.8 states the conjecture; we do not prove it.
2. **We do not prove the Langlands-RH proposition.** Theorem 7.4 states it; we do not inhabit it. RH is unproved classically, and we do not modify that fact.
3. **We do not internalise the liquid or solid axioms.** Theorem 6.1 records a partial obstruction to such internalisation in the standard HoTT–$(\infty, 1)$-topos correspondence; it does not rule it out in extended type theories.
4. **We do not prove the functional equation as condensed self-duality.** Theorem 7.1 states it; we do not establish it.

5. **We do not prove that the colimit-adele $\mathbb{A}_\mathbb{Q}$ of Theorem 4.1 is locally compact** or has the expected restricted-product topology. The topology is recovered only via the condensed embedding (Theorem 5.5), which uses the classical adele as input.

6. **We do not prove discreteness of $\mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}$ in the bare HoTT colimit.** Theorem 4.4 is conjectural.

7. **We do not extend the framework to function fields or to $\mathrm{GL}(n)$ over more general number fields.** Restriction to $\mathbb{Q}$ is for clarity; extending is an exercise that we expect to introduce no genuinely new HoTT difficulties beyond those already isolated.

8. **We do not address the directed-univalence aspects of Langlands functoriality.** OQ3 of Volume III addresses directed univalence; here we take only its discrete consequences.

The honest contribution of this paper is the formulation, the partial constructions ($\mathbb{Q}_p$, $\mathbb{A}_\mathbb{Q}$ as HIIT colimit, $\mathcal{E}_{\mathrm{cond}}$ as candidate), and the obstruction record.

10 Discussion

10.1 Why this matters

The arithmetic Langlands programme is widely seen as the principal arena where Riemann-zeta-type questions live [20]. Gate 5 and Gate 6 of the Volume II RH roadmap explicitly ask for a HoTT-native treatment of automorphic structures. Without identifying the topos, no progress on those gates is possible. The present paper records the first attempt: condensed mathematics is the clear candidate, but the obstruction we identify (Theorem 6.1) shows that pure HoTT is insufficient.

10.2 What infrastructure is missing

Three concrete pieces of infrastructure would unlock progress:

1. **Monoidal HoTT.** A type theory in which monoidal structures (in particular, $\mathbb{E}_n$-algebras) on the universe are part of the syntax, not external metatheory. Riehl's synthetic perspectives [17] and the Cisinski–Nguyen universal cocartesian fibration [16] suggest the shape of such a theory but it is not implemented.

2. **Locally compact HoTT.** A native treatment of locally compact Hausdorff spaces in HoTT, suitable for adelic objects. The cohesive HoTT of Schreiber and Shulman [13] is a partial step but does not yet handle the restricted-product topology.

3. **Directed Langlands.** Lifting the directed-univalence machinery of Volume III OQ3 [3, 15] from the discrete universe to the universe of automorphic representations. This is needed to express functoriality (a Langlands transfer of automorphic representations is a directed map) at the type-theoretic level.

10.3 Comparison with geometric Langlands

Gaitsgory et al. [14] prove geometric Langlands over function fields entirely within $(\infty, 1)$-categorical foundations, but classically. Their topos of choice is $\mathrm{IndCoh}(\mathrm{Bun}_G)$, an $(\infty, 1)$-topos of ind-coherent sheaves on the moduli stack of G-bundles. The arithmetic analogue, were it to exist, would replace Bun_G by an arithmetic-stack object built from $\mathbb{A}_{\mathbb{Q}}$. The condensed setting of Theorem 5.4 is a natural environment for such an object, but the precise statement of the arithmetic-stack analogue is open.

10.4 Comparison with pyknotic objects

Barwick–Haine [7] introduce pyknotic sets as condensed sets in an ∞-topos. The two frameworks (condensed and pyknotic) differ in size handling; the substantive content overlaps. Our candidate $\mathcal{E}_{\mathrm{cond}}$ would equally accept a pyknotic recasting. We chose condensed for concreteness, following Clausen–Scholze.

10.5 The Loeffler–Stoll comparison target

The Lean formalisation of ζ and the RH statement by Loeffler and Stoll [21] provides a classical comparison target. Our Theorem 7.4 is the HoTT-native analogue of their `RiemannHypothesisStatement`, lifted to the Langlands setting. A foundational comparison theorem (Volume III OQ4) would justify treating the two as "the same proposition".

10.6 Why $\mathcal{E}_{\mathrm{cond}}$ is a candidate, not the answer

The reader may reasonably ask: if $\mathcal{E}_{\mathrm{cond}}$ is an elementary $(\infty, 1)$-topos containing $\mathbb{A}_{\mathbb{Q}}$, $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, and (conjecturally) the smooth representations, why is it not the answer? Three reasons.

First, the smooth-representation reflectivity (Theorem 5.8) is conjectural. Without it, we do not have automorphic representations as objects of $\mathcal{E}_{\mathrm{cond}}$; we have only candidate ambient objects.

Second, the analytic structure — in particular, the topology making $\mathbb{A}_{\mathbb{Q}}$ locally compact and the diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ discrete — is recovered only via the embedding into condensed sets, not from the bare HoTT colimit. The HoTT-internal proof of discreteness (Theorem 4.4) is conjectural.

Third, the Langlands programme in its full form requires functoriality between automorphic-representation categories of different reductive groups: a Langlands-correspondence transfer is a directed map. The directed-univalence machinery needed to express such transfers natively (Volume III OQ3) is open.

A fourth, more methodological reason: a topos is not unique. Two distinct elementary $(\infty, 1)$-toposes can both contain a given collection of objects with all the right properties; they are related by geometric morphisms but not equal. Our claim "$\mathcal{E}_{\mathrm{cond}}$ is a candidate" should not be read as proposing a privileged choice; the natural setting may eventually prove to be a $\mathcal{E}$-localisation, a slice, or a different sheaf topos altogether.

10.7 Relationship to the ζ-flavoured RH gate

Volume II Part VI [2] gave three candidate definitions of ζ: as a HIIT, as the analytic limit of the Euler product, and as the unique meromorphic continuation. Each has a Gate 6 cousin via Mellin pairing with the trivial automorphic representation. The reduction Theorem 7.5 establishes the link, but does not pick out a single canonical definition. The advantage of working inside $\mathcal{E}_{\mathrm{cond}}$ is that all three definitions become objects of the same topos and their identification is a uniqueness-of-fixed-point argument internal to it. We expect this internalisation to be carried out in a follow-up paper of the present series.

10.8 Relationship to function-field Langlands

The geometric Langlands proof [14] works over function fields $k(X)$ for X a smooth projective curve over a finite field. The natural $(\infty, 1)$-topos there is $\mathrm{IndCoh}(\mathrm{Bun}_G(X))$, ind-coherent sheaves on the moduli of G-bundles. The arithmetic analogue would replace $\mathrm{Bun}_G(X)$ by an arithmetic-stack-like object built from $\mathbb{A}_{\mathbb{Q}}$.

The analytic-stacks lectures of Clausen–Scholze [6] sketch what such an arithmetic stack might look like in the condensed setting; they construct $\mathrm{Spec}(\mathbb{Z})$ as an analytic stack and discuss its global sections. Whether this analytic-stack-$\mathrm{Spec}(\mathbb{Z})$ admits a HoTT-native presentation, and whether the GL_n-bundle moduli over it is an internal type, are both open questions.

11 Conclusion

Volume III is the volume in which the HoTT-RH programme transitions from translation to research. OQ5 is the OQ in which that transition is most explicit: there is no natural elementary $(\infty, 1)$-topos for analytic Langlands in the literature, and no internalisation of condensed mathematics in HoTT.

This paper:

- Constructs $\mathbb{Q}_p$ as a HoTT HIIT (provable; Theorem 3.3).
- Constructs $\mathbb{A}_{\mathbb{Q}}$ as a HoTT colimit (provable, partial; Theorem 4.2).
- Sketches $\mathcal{E}_{\mathrm{cond}}$ as a candidate elementary $(\infty, 1)$-topos (provable elementarity; Theorem 5.3).
- Records a no-go-style obstruction to internalising solid/liquid axioms (open with partial proof; Theorem 6.1).
- States Gate 6 of the Volume II RH roadmap inside $\mathcal{E}_{\mathrm{cond}}$ (Theorem 7.4).
- Reduces Gate 6 to Volume II RH for ζ (provable, conditional; Theorem 7.5).

The contribution is the formulation, not the answer. Further progress requires the three pieces of infrastructure listed in Section 10: monoidal HoTT, locally compact HoTT, and directed Langlands. We expect each to be the subject of a future paper in the programme.

The companion code artifacts (a Haskell finite-place toy and Lean statement-only stubs) are described in Appendices A and B.

Acknowledgements. This paper inherits substantially from Volume II Parts II, IV, and V, and from the Volume III Knowledge Base.

A Haskell Implementation

We provide, in `src/oq5-infinity-topos-analytic-langlands/`, a finite-place toy of an automorphic-representation calculation. The toy implements:

- A finite-truncation $\mathbb{A}_{\mathbb{Q}}^{(N)} := \mathbb{Z}/N\mathbb{Z}$ of the adele ring, valid as an approximation when one cares only about residues modulo N.
- The group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(N)})$ as a finite group.
- An automorphic-form analogue: a $\mathbb{C}$-valued function on the double-coset space $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(N)})/K$ for $K_0(N)$ the standard congruence subgroup.
- Hecke operators T_p for $p \nmid N$, acting on automorphic forms.
- A QuickCheck property verifying commutativity $T_pT_q = T_qT_p$ for p, q distinct primes coprime to N.

The implementation is illustrative; it does not realise the full $(\infty, 1)$-topos structure (which is impossible at the level of finite computation). Its purpose is to provide a concrete handle on the objects whose home topos we are seeking.

A.1 Pseudo-code summary

The Haskell main loop computes T_2 and T_3 on a small basis of automorphic forms for GL_2 over $\mathbb{Z}/12\mathbb{Z}$ and verifies $T_2T_3 = T_3T_2$ for ten random basis vectors.

B Lean Stubs

In `lean/oq5-infinity-topos-analytic-langlands/`, we provide statement-only Lean files for:

1. `Padic.lean`: statement of Theorem 3.3 via Mathlib's existing `PadicNumbers`.
2. `Adele.lean`: statement of Theorem 4.2 as a colimit-of-rings statement.
3. `Condensed.lean`: a statement-only declaration of Theorem 5.3, citing the Clausen–Scholze condensed Lean library (in development).

4. `Gate6.lean`: statement-only of Theorem 7.4, with reduction Theorem 7.5 marked `sorry`.

These are intentionally stubs. Per Volume III's verdict contract [4], we declare the verdict as `incomplete` and we do not present the stubs as proofs.

References

Note on internal references. The companion documents [2, 3, 4] are part of the present *HoTT Foundations of Mathematics* programme. The prior published volume of the series [2] (Magneton Research, 2026) is cited for inherited theorems; the roadmap [3] and the knowledge base [4] are internal programme documents (compiled May 2026) released alongside the Volume III artifacts and cited here for traceability of stratification policy.

[1] Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* Institute for Advanced Study, 2013. arXiv:1308.0729.

[2] HoTT Foundations of Mathematics Programme, Volume II. *Cubical HIIT Cauchy Reals; Higher Contractibility of NNO; Directed Univalence; Langlands–Analytic Number Theory.* Magneton Research, 2026. (Published companion volume of this series; see paper-level references in the volume's six constituent files.)

[3] HoTT Foundations of Mathematics Programme. *Volume III Open Questions and Roadmap.* Internal source document, Magneton Research, 2026. (Released alongside the Volume III artifacts.)

[4] HoTT Foundations of Mathematics Programme. *Volume III Knowledge Base.* Internal compilation, Magneton Research, 2026-05-05. (Released alongside the Volume III artifacts; codifies the three-tier stratification policy of §1.3.)

[5] D. Clausen and P. Scholze. *Lectures on Condensed Mathematics.* Lecture notes, Bonn, 2019. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[6] D. Clausen and P. Scholze. *Analytic Stacks.* Lecture course, MPIM Bonn and IHES, 2023. `https://people.mpim-bonn.mpg.de/scholze/AnalyticStacks.html`.

[7] C. Barwick and P. Haine. *Pyknotic objects, I. Basic notions.* arXiv:1904.09966, 2019.

[8] J. Lurie. *Higher Topos Theory.* Annals of Mathematics Studies 170. Princeton University Press, 2009.

[9] J. Lurie. *Higher Algebra.* Preprint, 2017. `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[10] N. Rasekh. *A Theory of Elementary Higher Toposes.* arXiv:1805.03805, 2018.

[11] N. Rasekh. *Every Elementary Higher Topos has a Natural Number Object.* Theory and Applications of Categories, 37(13):337–377, 2021.

[12] M. Shulman. *All $(\infty, 1)$-toposes have strict univalent universes.* arXiv:1904.07004, 2019.

[13] M. Shulman. *Brouwer's fixed-point theorem in real-cohesive homotopy type theory.* Mathematical Structures in Computer Science, 28(6):856–941, 2018.

[14] D. Gaitsgory, S. Raskin, et al. *Proof of the geometric Langlands conjecture.* arXiv:2405.03599 and sequels, 2024.

[15] D. Gratzer, J. Weinberger, U. Buchholtz. *Directed univalence in simplicial homotopy type theory.* arXiv:2407.09146, 2024.

[16] D.-C. Cisinski and H. K. Nguyen. *The universal cocartesian fibration.* arXiv:2210.08945, 2022.

[17] E. Riehl. *Synthetic perspectives on spaces and categories.* arXiv:2510.15795, 2025.

[18] E. Riehl and D. Verity. *Elements of ∞-Category Theory.* Cambridge Studies in Advanced Mathematics 194. Cambridge University Press, 2022.

[19] J. Tate. *Fourier analysis in number fields, and Hecke's zeta-functions.* In: *Algebraic Number Theory* (Cassels and Fröhlich, eds.), Academic Press, 1967.

[20] R. P. Langlands. *Problems in the theory of automorphic forms.* Lecture Notes in Mathematics 170, Springer, 1970.

[21] D. Loeffler and M. Stoll. *Formalizing zeta and L-functions in Lean.* arXiv:2503.00959, 2025.

[22] J. N. Bernstein and A. V. Zelevinsky. *Representations of the group* $\mathrm{GL}(n, F)$ *where* F *is a non-Archimedean local field.* Russian Mathematical Surveys, 31(3):1–68, 1976.

[23] A. B. Booij. *Analysis in Univalent Type Theory.* PhD thesis, University of Birmingham, 2020.

[24] A. Vezzosi, A. Mörtberg, A. Abel. *Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types.* Proc. ACM Program. Lang. 3(ICFP):87, 2019.