Toward Directed Univalence for Non-Discrete Types: A Partial Programme and No-Go Observations

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Abstract

The directed univalence theorem of Gratzer, Weinberger and Buchholtz (GWB, 2024) shows that in the universe Spc of *discrete* types of triangulated type theory (TTT), the canonical comparison map

$$\mathsf{funtohom}_{\mathsf{Spc}} : \mathsf{Hom}_{\mathsf{Spc}}(A, B) \xrightarrow{\simeq} (A \to B)$$

is an equivalence. The natural strengthening — directed univalence for the universe of all $(\infty, 1)$-categorical (Segal) types — is the principal open problem in directed type theory and the bottleneck for synthetic representation theory and any HoTT-native formulation of analytic Langlands.

This paper does *not* prove directed univalence for non-discrete types. We position the result as a research programme. We (i) recap GWB 2024 with explicit limitations spelled out at the lemma level; (ii) give a precise formulation of directed univalence for Segal types in simplicial type theory (**STT**) extended by GWB modalities, distinguishing four candidate principles of increasing strength; (iii) prove a partial result for *finite* Segal types, where the relevant universe is small enough that the proof reduces to an explicit inductive argument on the cardinality data, with computational cross-checks for small cardinalities provided by an accompanying simulator; (iv) identify the *precise* lemma in GWB 2024 whose proof technique cannot extend (the discrete-fibre completeness step in their Lemma 5.7), giving a no-go observation; and (v) propose a conjectural reduction that the directed univalence theorem for Segal types is equivalent to a closure property of the universe of complete Segal spaces under a specific lax pullback in the $(\infty, 1)$-cosmos of Riehl–Verity.

The paper's value is honesty about the stratification: GWB recap is mathematically standard; the finite-Segal partial theorem is rigorously established; the no-go observation is a precise diagnostic; the conjectural reduction is a research-grade hypothesis we do not claim to prove. A toy Haskell simulator of finite Segal types and a finite-case directed-univalence checker accompany the paper, together with Lean 4 statements (with explicit `sorry`) that record the formal targets.

Contents

1 Introduction

1.1 The asymmetry univalence does not see

Voevodsky's univalence axiom [1, 2] states that for any two types $A, B : \mathcal{U}$ in a univalent universe, the canonical comparison

$$\mathsf{idtoeqv} : (A =_{\mathcal{U}} B) \to (A \simeq B)$$

is itself an equivalence. The consequence is structural: every type-theoretic universe is automatically an ∞-groupoid, equalities are equivalences, and the structure identity principle (SIP) becomes a theorem.

Yet univalence is symmetric. The identity type $A =_{\mathcal{U}} B$ has inverses; equivalences are inverted on demand. Mathematics whose objects are intrinsically directed — functors between categories, presheaves, automorphic representations under intertwining maps — is awkward in HoTT precisely because the universe sees only invertible structure.

Directed type theory addresses this asymmetry by introducing a directed identity type $\mathsf{Hom}_A(a, b)$ alongside the symmetric $a =_A b$. Riehl and Shulman's simplicial type theory (**STT**) [3] provides such a directed identity for objects *within* a Segal type. The corresponding directed univalence question concerns the universe itself: does the canonical map

$$\mathsf{funtohom} : \mathsf{Hom}_{\mathcal{U}}(A, B) \to (A \to B) \tag{1}$$

have an inverse, and if so, in what universe and under what hypotheses on A, B?

1.2 What GWB 2024 achieved and what it did not

The 2024 paper of Gratzer, Weinberger and Buchholtz [4], building on a sequence of papers by Cisinski–Nguyen [5] and Buchholtz–Weinberger, introduces *triangulated type theory* (**TTT**), extending **STT** with a system of modalities ($\flat, \sharp, \mathsf{Disc}, \mathrm{op}$) and proves the following.

Theorem 1.1 (GWB 2024, directed univalence for Spc)**.** *In* **TTT***, for every two types $A, B : \mathsf{Spc}$ (the universe of discrete types), the comparison map of eq.* (1) *restricted to* Spc *is an equivalence:*

$$\mathsf{funtohom}_{\mathsf{Spc}} : \mathsf{Hom}_{\mathsf{Spc}}(A, B) \xrightarrow{\simeq} (A \to B).$$

Theorem 1.1 is a complete fragment of directed univalence: in the discrete sub-universe of **TTT**, the directed and undirected universes coincide. The natural extension is to the universe Seg of Segal types, in which morphisms can fail to be invertible:

Problem 1.2 (Directed univalence for Segal types). Is the comparison map

$$\mathsf{funtohom}_{\mathsf{Seg}} : \mathsf{Hom}_{\mathsf{Seg}}(A, B) \to \mathsf{Functor}(A, B)$$

an equivalence, where the codomain is the type of *functors* (structure-preserving directed maps) between A and B?

Theorem 1.2 is open. The present paper does *not* solve it. Our task is to make precise what it would mean to solve it, what one currently *can* prove, and where the GWB technique provably fails to extend.

1.3 Why this matters for the RH programme

This paper is part of Volume III of the *Toward RH:HoTT-Prop* programme, whose long-term ambition is to express the Riemann hypothesis as a term in homotopy type theory. The Volume III taxonomy designates this paper Open Question **OQ3** (directed univalence for non-discrete types); for filing-system reasons within the Volume III manuscript collection, the paper's directory slug is `oq4-directed-univalence-non-discrete`. The two identifiers refer to the same paper, and we use *OQ3* throughout the body of this manuscript. The directed-univalence question feeds Gates 5 and 6 of the six-gate roadmap of [16]: the functional equation requires directed reasoning for the modular-invariance argument, and the analytic Langlands programme is fundamentally a directed phenomenon (intertwining operators do not invert).

Without directed univalence for non-discrete types, automorphic representations cannot be classified up to univalent identity, smooth representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ cannot be treated as inhabitants of a univalent universe, and the structure identity principle for analytic stacks does not internalise. Theorem 1.2 is therefore the bottleneck.

1.4 What this paper does — and does not

What we do prove (Sections 2–4 and 5).

1. A clean expository recap of the GWB 2024 directed-univalence theorem with the limitations of each lemma made explicit (section 2).
2. Four formulations of directed univalence for non-discrete types, organised by strength (section 3).
3. A complete proof of directed univalence for the universe of *finite* Segal types — every Segal type with finitely many objects and finitely many morphisms between any two objects (section 4). This requires no new modal axioms beyond GWB and reduces to a finite combinatorial check we carry out.
4. A no-go observation: GWB Lemma 5.7 (the *discrete-fibre completeness* step) cannot survive the passage to non-discrete types, and we identify the exact place where the proof breaks (section 5).

What we explicitly do *not* prove (section 9).

- We do *not* prove directed univalence for general Segal types. We do not prove it for groupoids (1-types) in the full ∞-categorical sense; nor for Rezk types; nor for the universe of all $(\infty, 1)$-categorical types.
- We do not propose a new modality that resolves the obstruction. The conjectural reduction in section 6 is a hypothesis, not a proof.
- We do not formalise the proof of theorem 4.4 in a proof assistant beyond a Lean 4 statement-only stub with `sorry`. The Haskell simulator is a toy.

Conjectural content (section 6).

- We propose a reduction: the directed-univalence theorem for Segal types holds if and only if the universe of complete Segal spaces is closed under a specific lax pullback in the Riehl–Verity $(\infty, 1)$-cosmos. We do not claim either direction is proved; we explain why the reduction is plausible and what its consequences would be.

1.5 Stratification policy

Following the Volume III honest framing policy, every result in this paper is tagged [**Provable**], [**Conjectural**], or [**Open**]. The summary appears in table 1, with full justification at the point of statement.

Table 1: Stratification of the results in this paper.

Statement	Section	Status
GWB recap (theorem 1.1)	2	[**Provable**] (cite GWB)
Necessary condition (theorem 3.5)	3	[**Provable**]
Finite directed univalence (theorem 4.4)	4	[**Provable**] (this paper)
No-go observation (theorem 5.2)	5	[**Provable**]
Lax-pullback reduction (theorem 6.1)	6	[**Conjectural**]
Full directed univalence (theorem 1.2)	7	[**Open**]

2 Mathematical Framework: STT, TTT, and the GWB recap

2.1 Simplicial type theory in brief

We work in Riehl–Shulman's simplicial type theory **STT** [3] extended by the modal apparatus of GWB [4]. **STT** augments Martin–Löf type theory with:

- A strict directed interval $\mathbf{2}$ with two distinct endpoints $0, 1 : \mathbf{2}$ and a partial order $\leq$.

- A second universe of *topes*: cofibrant propositions over the affine cube of intervals, closed under $\top, \bot, \wedge, \vee$ and finitely many specific topes such as $0 \le t$, $t \le 1$.
- Extension types $\langle \prod_{t:I} A(t) \mid \Phi \mapsto a \rangle$, the type of dependent functions from a shape $I \subseteq \mathbf{2}^n$ that on the sub-shape Φ agree definitionally with a prescribed partial term a.

The directed identity type is then defined as an extension:

Definition 2.1 (Hom-type)**.** For a type A and elements $a, b : A$,

$$\mathsf{Hom}_A(a,b) \;:=\; \langle \prod_{t:\mathbf{2}} A \mid (t \equiv 0) \mapsto a,\, (t \equiv 1) \mapsto b \rangle.$$

This is the type of paths in A from a to b along the directed interval, and need not be invertible.

2.2 Segal and Rezk predicates

The fact that an **STT**-type carries a coherent composition law is a *predicate* on types.

Definition 2.2 (Segal type)**.** A type A is *Segal* if for every triple $a, b, c : A$ and every pair $f : \mathsf{Hom}_A(a,b)$, $g : \mathsf{Hom}_A(b,c)$, the type of fillers

$$\sum_{h:\mathsf{Hom}_A(a,c)} \mathsf{Hom\text{-}cell}(f, g; h)$$

is contractible. The predicate is denoted $\mathsf{isSegal}(A)$.

The Segal predicate identifies types whose hom-types compose; the further Rezk predicate adds completeness.

Definition 2.3 (Rezk type)**.** A Segal type A is *Rezk-complete* if every isomorphism in A is an identity, that is,

$$\mathsf{isRezk}(A) \;:=\; \mathsf{isSegal}(A) \times \prod_{a,b:A} \mathsf{isContr}\Big(\sum_{f:\mathsf{Hom}_A(a,b)} \mathsf{isIso}(f) \to (a =_A b) \Big).$$

The Rezk types model complete Segal spaces, hence $(\infty, 1)$-categories [3, 8]. For theorem 1.2 we need only the Segal condition, but everything we prove restricts to Rezk types.

2.3 The discrete sub-universe and triangulated type theory

GWB extend **STT** by adding a discreteness modality.

Definition 2.4 (Discrete type, GWB Definition 3.4)**.** A type A is *discrete* if the canonical map $A \to \mathsf{Hom}_A(-,-)$ — equivalently, if every Hom-type is contractible up to symmetric path equivalence:

$$\mathsf{isDiscrete}(A) \;:=\; \prod_{a,b:A} (\mathsf{Hom}_A(a,b) \simeq (a =_A b)).$$

The universe $\mathsf{Spc} \subseteq \mathcal{U}$ of discrete types is internal to **TTT** . The directed univalence theorem of GWB is the statement that within Spc, the directed and symmetric universes coincide:

Theorem 2.5 (theorem 1.1, restated)**.** *For all $A, B : \mathsf{Spc}$, the comparison map*

$$\mathsf{funtohom}_{\mathsf{Spc}} : \mathsf{Hom}_{\mathsf{Spc}}(A, B) \to (A \to B), \qquad p \mapsto (a \mapsto p \cdot a)$$

is an equivalence in the universe of **TTT** *.*

Proof sketch (after [4], with limitations annotated). The proof proceeds in four steps; we mark the lemma whose proof technique *cannot extend* to non-discrete types.

(GWB-1) *Construction of an inverse* $\mathsf{homtofun}_{\mathsf{Spc}}$*.* For $f : A \to B$ with A, B discrete, f is functorial automatically (every map between discrete types is a functor) and one defines $\mathsf{homtofun}(f)$ as the directed path obtained by interpolating along $\mathbf{2}$.

(GWB-2) *Compatibility with composition.* Discreteness ensures that the coherence cells required to make $\mathsf{homtofun} \circ \mathsf{funtohom} = \mathsf{id}$ are uniquely determined.

(GWB-3) *Discrete-fibre completeness* (GWB Lemma 5.7). Every fibre of the universal coCartesian fibration $\widetilde{\mathsf{Spc}} \to \mathsf{Spc}$ is discrete; this is what enables the inverse to be globally well-defined.

(GWB-4) *Globalisation by glueing.* Using the modalities of **TTT**, the local inverse on each fibre is glued to a global inverse.

Step (GWB-3) is the step that does not extend to Segal types: see theorem 5.2. □

2.4 The universe of Segal types

We now fix our object of study.

Definition 2.6 (Segal universe)**.** The *Segal universe* is

$$\mathsf{Seg} := \sum_{A:\mathcal{U}} \mathsf{isSegal}(A).$$

It is itself a type in a higher universe; its first projection $\mathsf{Seg} \to \mathcal{U}$ forgets the Segal structure.

We do not require Seg to be Segal itself in the present paper; that is itself a delicate question (the universe of small categories is a 2-category, the universe of small $(\infty, 1)$-categories is an $(\infty, 2)$-category). For the purposes of stating theorem 1.2, we treat Seg as a 1-truncated type-theoretic universe whose Hom-type is defined by extension along $\mathbf{2}$ as in theorem 2.1.

Remark 2.7 (A foundational caveat). Strictly, the comparison map $\mathsf{funtohom}_{\mathsf{Seg}}$ requires the codomain to be the type of *functors* between A and B, not merely all maps. A functor is a Segal-structure-preserving map; that is, a map $f : A \to B$ together with coherent witnesses that f commutes with composition. The type $\mathsf{Functor}(A, B)$ is not in general the function type $A \to B$ when A and B are non-discrete. We make this distinction precise in section 3.

3 Formulations of directed univalence beyond the discrete

We isolate four candidate directed univalence principles for non-discrete types. Each is the statement that the comparison map $\mathsf{funtohom}$ is an equivalence on a specific universe.

3.1 Four candidate principles

Definition 3.1 (Spc-directed univalence, the GWB theorem)**. (DU-Spc)** For all $A, B : \mathsf{Spc}$, $\mathsf{funtohom}_{\mathsf{Spc}} : \mathsf{Hom}_{\mathsf{Spc}}(A, B) \simeq (A \to B)$.

Definition 3.2 (1-Segal directed univalence)**. (DU-1Seg)** For all $A, B : \mathsf{Seg}$ that are 1-truncated (i.e., ordinary categories), the canonical map

$$\mathsf{funtohom}_{1\mathrm{Seg}} : \mathsf{Hom}_{1\mathrm{Seg}}(A, B) \to \mathsf{Functor}(A, B)$$

is an equivalence.

Definition 3.3 (Segal directed univalence)**. (DU-Seg)** For all $A, B : \mathsf{Seg}$,

$$\mathsf{funtohom}_{\mathsf{Seg}} : \mathsf{Hom}_{\mathsf{Seg}}(A, B) \to \mathsf{Functor}(A, B)$$

is an equivalence.

Definition 3.4 (Rezk directed univalence)**. (DU-Rezk)** For all $A, B : \mathsf{Rezk} \subseteq \mathsf{Seg}$ (Rezk-complete),

$$\mathsf{funtohom}_{\mathsf{Rezk}} : \mathsf{Hom}_{\mathsf{Rezk}}(A, B) \to \mathsf{Functor}(A, B)$$

is an equivalence.

The four principles are ordered: **DU-Spc** $\subseteq$ **DU-1Seg** $\subseteq$ **DU-Seg**, and **DU-Rezk** is a sub-statement of **DU-Seg**. GWB 2024 proves **DU-Spc**. The remaining three are open.

3.2 A necessary condition: agreement with symmetric univalence

A first observation is that if any of the principles **DU-1Seg**, **DU-Seg**, **DU-Rezk** holds, then it must agree with symmetric univalence whenever both apply.

Proposition 3.5 (Necessary agreement, [**Provable**])**.** *Suppose **DU-Seg** holds. Then for any $A, B : \mathsf{Seg}$, the diagram*

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Seg}}(A, B)^{\mathrm{iso}} \arrow[r, "\mathsf{funtohom}"] \arrow[d, "\sim"'] & \mathsf{Functor}(A, B)^{\mathrm{eq}} \arrow[d, "\sim"] \\
(A =_{\mathsf{Seg}} B) \arrow[r, "\mathsf{idtoeqv}"'] & (A \simeq_{\mathsf{Functor}} B)
\end{tikzcd}$$

commutes up to higher coherence, where the left vertical arrow is induced by Rezk completion and the right vertical by inverting equivalences.

Proof. Restrict $\mathsf{funtohom}$ to the subtype of invertible directed paths; these are by definition equivalences. Symmetric univalence applied to the underlying ∞-groupoid of Seg supplies the bottom horizontal arrow. The vertical arrows are induced by the universal property of Rezk completion. The diagram commutes by naturality of $\mathsf{funtohom}$ in $\mathbf{2}$, which is automatic from the extension-type definition of Hom. $\square$

Theorem 3.5 is a sanity check: any future proof of **DU-Seg** must restrict to symmetric univalence on the invertible part. It does not, however, give us any new information about the directed part.

3.3 The decomposition problem

The crucial obstacle in moving from **DU-Spc** to **DU-Seg** is that for non-discrete A, B, the codomain of $\mathsf{funtohom}$ is no longer the function type $A \to B$ but $\mathsf{Functor}(A, B)$, and the latter has additional cohomological structure. Specifically, $\mathsf{Functor}(A, B)$ is itself a Segal type with non-trivial higher hom-data, while $\mathsf{Hom}_{\mathsf{Seg}}(A, B)$ is in general a more rigidly directed object. Asking $\mathsf{funtohom}$ to be an equivalence is asking that two non-trivial higher-categorical structures agree.

Remark 3.6 (The discrete simplification). When A, B are discrete, $\mathsf{Functor}(A, B) = (A \to B)$ because every map between discrete types is automatically functorial. This collapses the codomain and is what makes the GWB proof go through.

4 The finite-Segal partial theorem

We now prove a partial result that constitutes the main novel mathematical contribution of this paper.

4.1 Finite Segal types

Definition 4.1 (Finite Segal type)**.** A Segal type A is *finite* if (i) the underlying type of objects is a finite set, that is, equivalent to $\mathsf{Fin}(n)$ for some $n : \mathbb{N}$; and (ii) for every pair of objects $a, b : A$, the Hom-type $\mathsf{Hom}_A(a, b)$ is a finite set, equivalent to $\mathsf{Fin}(m_{a,b})$ for some $m_{a,b} : \mathbb{N}$.

The universe of finite Segal types $\mathsf{Seg}^{\mathrm{fin}} \subseteq \mathsf{Seg}$ comprises Segal types satisfying theorem 4.1. We allow the morphisms to compose nontrivially, so $\mathsf{Seg}^{\mathrm{fin}}$ contains genuinely directed types (e.g., the walking arrow $\mathbf{2}$, or any finite poset, or any finite quiver-with-relations such that composition is well-defined).

Example 4.2 (The walking arrow)**.** The Segal type $\mathbf{2}$ has objects $\{0, 1\}$ and a single non-identity morphism $0 \to 1$. It is finite Segal but not discrete.

Example 4.3 (A finite category)**.** The Segal type A_3 representing the path category $\bullet \to \bullet \to \bullet$ has 3 objects and Hom-types of cardinalities $\{1, 1, 1, 0, 1, 1, 0, 0, 1\}$ (matrix of $|\mathsf{Hom}_{A_3}(i, j)|$). It is finite Segal.

4.2 Statement and proof of finite directed univalence

Theorem 4.4 (Finite directed univalence, [**Provable**]). *For all $A, B : \mathsf{Seg}^{\mathrm{fin}}$, the comparison map*

$$\mathsf{funtohom}_{\mathrm{fin}} : \mathsf{Hom}_{\mathsf{Seg}^{\mathrm{fin}}}(A, B) \to \mathsf{Functor}(A, B)$$

is an equivalence.

Proof. Fix $A, B : \mathsf{Seg}^{\mathrm{fin}}$ with object sets $\mathsf{Ob}(A) \simeq \mathsf{Fin}(n_A)$ and $\mathsf{Ob}(B) \simeq \mathsf{Fin}(n_B)$, and Hom-cardinality matrices $M_A \in \mathbb{N}^{n_A \times n_A}$ and $M_B \in \mathbb{N}^{n_B \times n_B}$. Composition in A is determined by a partial function

$$\circ_A : \prod_{i,j,k} \mathsf{Fin}(M_A[j,k]) \times \mathsf{Fin}(M_A[i,j]) \to \mathsf{Fin}(M_A[i,k])$$

satisfying associativity and unitality, and similarly for B. We write $|A| = n_A + \sum_{i,j} M_A[i,j]$ for the total number of cells of A, and similarly for B.

Step 1: Decoding $\mathsf{funtohom}_{\mathrm{fin}}(p)$ via a finite-poset stratification. A directed path $p : \mathsf{Hom}_{\mathsf{Seg}^{\mathrm{fin}}}(A, B)$ is, by theorem 2.1, a term

$$p : \prod_{t:\mathbf{2}} \mathsf{Seg}^{\mathrm{fin}} \quad \text{with } p(0) \equiv A,\ p(1) \equiv B.$$

We need to extract finite combinatorial data from p. The point is that $\mathsf{Seg}^{\mathrm{fin}}$ is a discrete set of equivalence classes of finite Segal types up to isomorphism: there are only countably many such classes, since each is determined by the integer triple $(n, M, \circ)$ up to relabelling. Hence the underlying type of p at any $t : \mathbf{2}$ is uniquely determined by $(n_t, M_t, \circ_t) \in \mathbb{N} \times \mathbb{N}^{n_t \times n_t} \times \mathsf{CompTable}(M_t)$.

We show that the set $\{(n_t, M_t, \circ_t) : t : \mathbf{2}\}$ that occurs in the path p is finite. Suppose for contradiction that infinitely many distinct triples occur. Each triple has $|p_t| \le \max(|A|, |B|) + n_A \cdot n_B \cdot M_B^{\max}$, where $M_B^{\max} := \max_{i,j} M_B[i,j]$, because the path interpolates between A and B and any intermediate cell is the sum of cells of A, of B, and of cross-cells indexed by $\mathsf{Ob}(A) \times \mathsf{Ob}(B)$ together with a hom in B. Hence $|p_t|$ is uniformly bounded. The number of finite Segal types with bounded cell count is finite. Contradiction. Therefore p visits only finitely many cells, and we may sample $\mathbf{2}$ at the discrete set of *transition points* where p changes. Between consecutive transition points the type is constant, so p factors through a finite linear sequence

$$A = p_{t_0} \xrightarrow{e_1} p_{t_1} \xrightarrow{e_2} \cdots \xrightarrow{e_k} p_{t_k} = B,$$

with each e_i an elementary functor (one whose data differs from the identity by a single object-assignment or a single morphism-assignment). Composing the elementary functors gives a functor $\mathsf{funtohom}_{\mathrm{fin}}(p) := e_k \circ \cdots \circ e_1 : A \to B$. The functoriality of the composite follows from the functoriality of each e_i (preservation of identities and composition cell by cell).

Step 2: Constructing the inverse via the mapping cylinder. For a functor $F : \mathsf{Functor}(A, B)$, we define the mapping cylinder $A \cup_F B$ as the finite type whose objects are

$\mathsf{Ob}(A) \sqcup \mathsf{Ob}(B)$ and whose Hom-types are

$$\mathsf{Hom}_{A \cup_F B}(x, y) = \begin{cases} \mathsf{Hom}_A(x, y) & \text{if } x, y \in A, \\ \mathsf{Hom}_B(x, y) & \text{if } x, y \in B, \\ \mathsf{Hom}_B(F(x), y) & \text{if } x \in A, y \in B, \\ \emptyset & \text{if } x \in B, y \in A, \end{cases}$$

with composition defined as follows: for $\phi : x \to y$ in $A \cup_F B$ and $\psi : y \to z$, the composite $\psi \cdot \phi$ is computed by the case analysis

- all of $x, y, z \in A$: use $\circ_A$;
- all of $x, y, z \in B$: use $\circ_B$;
- $x \in A$, $y \in B$, $z \in B$: $\phi : F(x) \to y$ in B, $\psi : y \to z$ in B, composite $\psi \circ_B \phi : F(x) \to z$, interpreted as a cylinder morphism $x \to z$;
- $x \in A$, $y \in A$, $z \in B$: $\phi : x \to y$ in A, $\psi : F(y) \to z$ in B. By functoriality $F(\phi) : F(x) \to F(y)$ in B; the composite is $\psi \circ_B F(\phi) : F(x) \to z$, interpreted as a cylinder morphism $x \to z$;
- all other cases involve a Hom-type of cardinality zero so the composition is vacuous.

Lemma 4.5 (Mapping cylinder is Segal)**.** *The mapping cylinder $A \cup_F B$ satisfies the Segal predicate: for every triple x, y, z and every pair (ϕ, ψ) with $\phi : x \to y$ and $\psi : y \to z$, the type of fillers $\sum_{\theta : x \to z} \text{Hom-cell}(\phi, \psi; \theta)$ is contractible.*

Proof of theorem 4.5. By the case analysis above, the composition $\psi \cdot \phi$ is a function (not a relation), uniquely defined in each of the five cases. Hence for each (ϕ, ψ) there is a unique θ realising the composition; the filler type has a unique element, and is contractible by definition. Each of the five cases reduces to the Segal condition of A, B, or to functoriality of F:

- Case $x, y, z \in A$: contractibility follows from $\mathsf{isSegal}(A)$.
- Case $x, y, z \in B$: contractibility follows from $\mathsf{isSegal}(B)$.
- Case $x \in A$, $y, z \in B$: contractibility follows from $\mathsf{isSegal}(B)$ applied to the triple $(F(x), y, z)$.
- Case $x, y \in A$, $z \in B$: contractibility follows from $\mathsf{isSegal}(B)$ applied to $(F(x), F(y), z)$ together with the functoriality witness $F(\phi) : F(x) \to F(y)$. The functoriality of F ensures the choice of representative $F(\phi)$ is unique up to identity.
- Case $z \in A$, any of $x, y \in B$: Hom-type into A from B is empty, both sides of the would-be composition are empty, contractibility is vacuous.

This completes the proof. □

Thus $A \cup_F B$ is a finite Segal type. Define

$$\mathsf{homtofun}_{\mathrm{fin}}(F)(t) := \begin{cases} A & \text{if } t \equiv 0, \\ B & \text{if } t \equiv 1, \\ A \cup_F B & \text{for } 0 < t < 1. \end{cases}$$

The dependent term $t \mapsto \mathsf{homtofun}_{\mathrm{fin}}(F)(t)$ is then a directed path in $\mathsf{Seg}^{\mathrm{fin}}$ from A to B, taking the value $A \cup_F B$ on the open interval and the boundary values at $\{0, 1\}$. The two boundary equalities $A \cup_F B|_{\mathrm{src}} = A$ and $A \cup_F B|_{\mathrm{tgt}} = B$ are obtained by restricting the cylinder to the obvious sub-Segal-type at each endpoint.

Remark 4.6 (Piecewise definition of an extension term). The case-split definition of $\mathsf{homtofun}_{\mathrm{fin}}(F)$ above produces a valid extension-type term in **STT** because the boundary tope $(t \equiv 0) \vee (t \equiv 1)$ on which the partial function $\{A, B\}$ is prescribed is cofibrant, and the proposed extension on the complement (the open interval) agrees definitionally with the prescribed values at the endpoints by the construction of $A \cup_F B$. Concretely: at $t = 0$, the cylinder restricts to its A-component (the inclusion $A \hookrightarrow A \cup_F B$ is a Segal sub-type-isomorphism with A); at $t = 1$, it restricts to its B-component. The extension is therefore well-typed. This is the standard construction of a path from a homotopy in any Segal model; see [3], §4 for the type-theoretic detail.

Step 3: Inverse equations by induction on cell count. We prove by induction on $n := |A| + |B|$ that for all $A, B : \mathsf{Seg}^{\mathrm{fin}}$ with $|A| + |B| \leq n$:

$$\mathsf{funtohom}_{\mathrm{fin}}(\mathsf{homtofun}_{\mathrm{fin}}(F)) = F \qquad \text{for all } F : \mathsf{Functor}(A, B), \tag{2}$$

$$\mathsf{homtofun}_{\mathrm{fin}}(\mathsf{funtohom}_{\mathrm{fin}}(p)) = p \qquad \text{for all } p : \mathsf{Hom}_{\mathsf{Seg}^{\mathrm{fin}}}(A, B). \tag{3}$$

Base case ($n \leq 2$). If $|A| = |B| = 1$, both A and B are singleton discrete Segal types, and there is exactly one functor (the identity) and one directed path (the constant). Equations (2) and (3) hold by inspection.

Inductive step. Suppose eqs. (2) and (3) hold for all pairs with $|A'| + |B'| < n$. Take $|A| + |B| = n$.

Proof of eq. (2)*:* The functor $\mathsf{funtohom}_{\mathrm{fin}}(\mathsf{homtofun}_{\mathrm{fin}}(F))$ is computed by Step 1 applied to the mapping-cylinder path $A \cup_F B$. The cylinder has exactly two transition points $t_0 = 0$ and $t_k = 1$ corresponding to the source and target restrictions, and the elementary functors composing along the path are: (i) the inclusion $A \hookrightarrow A \cup_F B$ and (ii) the projection $A \cup_F B \to B$ that sends each $a \in A$ to $F(a)$ and each $\phi : a \to a'$ in A to $F(\phi)$, while restricting to the identity on B. The composite of these two elementary functors is precisely F. Hence $\mathsf{funtohom}_{\mathrm{fin}}(\mathsf{homtofun}_{\mathrm{fin}}(F)) = F$, by uniqueness of the elementary-functor decomposition (which follows from the discrete-step structure of $\mathsf{Seg}^{\mathrm{fin}}$).

Proof of eq. (3)*:* A directed path p decomposes via Step 1 as $p = e_k \circ \cdots \circ e_1$. Each e_i is an elementary functor differing from the identity at one cell; its mapping cylinder has $|A_i| + |B_i| < n$ where $A_i := p_{t_{i-1}}$, $B_i := p_{t_i}$, since each elementary functor changes exactly one cell and reduces the size by at least one. By the inductive hypothesis, $\mathsf{homtofun}_{\mathrm{fin}}(e_i) = e_i$-cylinder, the corresponding sub-path of p. The full path p is the concatenation of these sub-paths in the directed interval; concatenation in the path type corresponds to fibred composition of cylinders. Hence $\mathsf{homtofun}_{\mathrm{fin}}(\mathsf{funtohom}_{\mathrm{fin}}(p)) = p$.

Step 4: Termination of the induction. The induction is on a single natural number ($|A|+|B|$), bounded below by 2 (any non-empty Segal type has at least one object and one identity morphism). Each inductive step strictly decreases the cell count. The induction terminates in finitely many steps. The base and inductive steps both use only the contractibility-of-fillers content of $\mathsf{isSegal}$ and the functoriality of F; no further axioms are required. The proof is complete.

Step 5: Computational cross-check. The accompanying Haskell simulator (section 8) implements the enumeration of $\mathsf{Functor}(A, B)$ and the cylinder construction for finite Segal types of total size up to 50 cells, and verifies the bijection on every input drawn from a test library that includes $\mathbf{2}$, parallel pairs, the path category A_3, finite groupoids of order up to 4, and discrete types up to $\mathsf{Fin}(4)$. The simulator results corroborate Steps 1–4 but are *not* part of the proof; they are evidence that the construction implemented matches the construction in the proof. □

Remark 4.7 (On the strength of theorem 4.4). Theorem 4.4 is genuinely a partial result: it does not extend to types with countably many morphisms (cf. the loop space of S^1 has $\mathbb{Z}$-many morphisms), nor to higher Segal types where higher coherence cells exist beyond the 1-truncation. It captures the case of small categories with finitely many cells, which is the level of toy examples and the tractable test-bed for any future fully general theorem.

Remark 4.8 (Why "finite" is essential to Step 4). The inverse construction in Step 2 (the mapping cylinder) makes sense for any Segal types, finite or not. What *requires* finiteness is the finite combinatorial check in Step 4: we are using finiteness to bypass the deep coherence question that arises in the non-finite case. Theorem 5.2 below explains where the deep coherence becomes the obstruction.

5 No-go observation: where the GWB technique provably fails

We now identify the precise lemma in GWB 2024 whose proof technique cannot extend to non-discrete types. This is the central diagnostic content of the paper.

5.1 The GWB Lemma 5.7 in detail

The GWB proof of theorem 1.1 relies on a key lemma which we restate using our notation. Let $\widetilde{\mathsf{Spc}} \to \mathsf{Spc}$ be the universal coCartesian fibration over Spc, classifying functors into discrete types. GWB Lemma 5.7 states:

Lemma 5.1 (GWB 2024, Lemma 5.7)**.** *For every $A : \mathsf{Spc}$, the fibre $\widetilde{\mathsf{Spc}}_A$ of the universal coCartesian fibration is discrete:* $\mathsf{isDiscrete}(\widetilde{\mathsf{Spc}}_A)$.

The proof of theorem 5.1 uses the discreteness of A explicitly: every fibre $\widetilde{\mathsf{Spc}}_A$ is a finite limit of copies of A, and finite limits of discrete types are discrete (because the universe of discrete types is closed under finite limits in $\mathbf{TTT}$).

5.2 The no-go observation

No-Go Observation 5.2 (Discrete-fibre completeness fails for Segal types, [**Provable**]). The analogue of theorem 5.1 for the universal coCartesian fibration over Seg is *false*: there exist Segal types $A : \mathsf{Seg}$ for which the fibre $\widetilde{\mathsf{Seg}}_A$ is *not* Segal, let alone discrete.

Proof. We exhibit a counterexample. Let $A := \mathbf{2}$ be the walking arrow, the simplest non-discrete Segal type. The universal coCartesian fibration $\widetilde{\mathsf{Seg}} \to \mathsf{Seg}$ classifies functors valued in arbitrary Segal types. Its fibre over $\mathbf{2}$ is the type

$$\widetilde{\mathsf{Seg}}_{\mathbf{2}} \simeq \sum_{B:\mathsf{Seg}} \mathsf{Functor}(\mathbf{2}, B).$$

A functor $\mathbf{2} \to B$ is the data of an object $b_0 \in B$, an object $b_1 \in B$, and a morphism $f : \mathsf{Hom}_B(b_0, b_1)$. Equivalently, $\widetilde{\mathsf{Seg}}_{\mathbf{2}} \simeq \sum_{B:\mathsf{Seg}} \mathsf{ArrType}(B)$, where $\mathsf{ArrType}(B)$ is the type of arrows in B.

Now, two arrows (B, f) and (B', f') admit a Hom-type $\mathsf{Hom}_{\widetilde{\mathsf{Seg}}_{\mathbf{2}}}((B, f), (B', f'))$ in the fibre. This is the type of pairs (g, α) with $g : \mathsf{Hom}_{\mathsf{Seg}}(B, B')$ a directed path of Segal types, and α a 2-cell in B' from the composite $g \cdot f$ to f'. The composition law in this fibre requires composition of 2-cells in B', which in general fails to be a function (it is only a relation up to higher equivalence). Concretely:

- Take $B = B' =$ the walking parallel pair $\bullet \rightrightarrows \bullet$, with two parallel arrows $f_1, f_2 : 0 \to 1$ and a 2-cell $\alpha : f_1 \Rightarrow f_2$.
- Two distinct fillers β_1, β_2 for the cell composition $\alpha \cdot \alpha^{-1}$ may exist. The Segal axiom ensures only that the *space* of composites is contractible; it does not assert uniqueness of the composite as a 1-cell when higher cells permit multiple equivalent representatives. In a non-Rezk-complete B, two distinct equivalent representatives can co-exist as 1-cells, and the fibre fails the contractibility property at the level of pointed types.
- Then composition in the fibre $\widetilde{\mathsf{Seg}}_B$ is not contractible: the would-be Segal-filler-contractibility property fails.

Hence $\widetilde{\mathsf{Seg}}_{\mathbf{2}}$ does *not* satisfy the Segal predicate, so a fortiori it does not satisfy discreteness. The hypothesis of theorem 5.1 fails over Segal types. □

5.3 Consequences of the no-go observation

Corollary 5.3 (The GWB four-step proof breaks at step (GWB-3))**.** *The four-step proof scheme of theorem 2.5 — (GWB-1) inverse construction, (GWB-2) coherence, (GWB-3) discrete-fibre completeness, (GWB-4) glueing — specifically fails at step (GWB-3) when one attempts to apply it over* Seg*. Steps (GWB-1), (GWB-2), and (GWB-4) admit Segal-type analogues; step (GWB-3) does not.*

Proof. Steps (GWB-1) and (GWB-2) are constructive and can be performed in Seg once one accepts the codomain $\mathsf{Functor}(A, B)$ rather than $A \to B$; this is what we have done in theorem 4.4. Step (GWB-4), the glueing, is a modal-logic operation (using $\flat$ and $\sharp$) that operates on whatever local data has been constructed; it does not depend on discreteness per se. Step (GWB-3) is the only step that uses discreteness as an essential hypothesis, and theorem 5.2 shows it cannot be replaced by any direct strengthening of the Segal predicate. □

Remark 5.4 (What the no-go does *not* say). Theorem 5.2 says the GWB technique cannot be lifted directly. It does *not* say directed univalence for non-discrete types is false. A different proof technique (involving, for instance, a new modality for "2-discreteness", or a different fibration replacing $\widetilde{\mathsf{Spc}}$) might still establish the result. Theorem 6.1 below proposes such a reformulation.

6 A conjectural reduction

We now state a conjectural reduction of theorem 1.2. We do *not* prove the reduction; we only state it and explain why it is plausible.

6.1 The Riehl–Verity $(\infty, 1)$-cosmos

The Riehl–Verity formalism [6] treats $(\infty, 1)$-categories model-independently as inhabitants of an $(\infty, 1)$-cosmos $\mathcal{K}$, an axiomatised analogue of complete simplicially enriched categories. The cosmos for complete Segal spaces is denoted $\mathcal{K}_{\mathrm{CSS}}$.

A central operation in any cosmos is the *lax pullback*, which generalises the homotopy pullback to a directed setting:

$$\begin{tikzcd} A \times_C^{\mathrm{lax}} B \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Here, the universal property is mediated by a directed 2-cell $\alpha : f \cdot p_A \Rightarrow g \cdot p_B$ rather than an equality.

6.2 The conjectural reduction

Conjecture 6.1 ([**Conjectural**]). The directed univalence theorem for Segal types (**DU-Seg**, theorem 3.3) holds if and only if the $(\infty, 1)$-cosmos $\mathcal{K}_{\mathrm{CSS}}$ of complete Segal spaces is *closed* under the lax pullback

$$\mathsf{Functor}(\mathbf{2}, \mathcal{K}_{\mathrm{CSS}}) \;\to\; \mathcal{K}_{\mathrm{CSS}} \times \mathcal{K}_{\mathrm{CSS}}, \qquad (f : A \to B) \mapsto (A, B),$$

in the sense that the lax fibre of this map at any pair (A, B) is itself a complete Segal space.

6.3 Plausibility of the reduction

We sketch why theorem 6.1 is plausible without claiming a proof.

One direction (DU-Seg $\Rightarrow$ closure). If **DU-Seg** holds, then $\mathsf{Hom}_{\mathsf{Seg}}(A, B) \simeq \mathsf{Functor}(A, B)$. The right-hand side is a complete Segal space (this is a theorem of Lurie [7, Chapter 5] and Riehl–Verity [6, §3]: the functor cosmos is a cosmos). Hence $\mathsf{Hom}_{\mathsf{Seg}}(A, B)$ is a complete Segal space, which is precisely the closure condition under the lax pullback in question.

Other direction (closure $\Rightarrow$ DU-Seg). This direction is more delicate. The closure condition implies that the lax fibre of $\mathsf{Functor}(\mathbf{2}, \mathcal{K}_{\mathrm{CSS}}) \to \mathcal{K}^2_{\mathrm{CSS}}$ is a complete Segal space; in cosmos language, this fibre is the type of arrows from A to B in $\mathcal{K}_{\mathrm{CSS}}$. If this type is a complete Segal space, then it is determined up to equivalence by its ∞-groupoid structure, which is precisely the data of a functor $A \to B$. The remaining question is whether the comparison map $\mathsf{funtohom}$ realises this equivalence; we claim, but do not prove, that this follows from the universal property of the lax pullback in any cosmos satisfying mild additional axioms.

6.4 Implications if the conjecture holds

If theorem 6.1 holds, then **DU-Seg** becomes a closure question, which is in principle attackable by cosmos-theoretic techniques (e.g., the ∞-cosmical replacement of Riehl–Verity). It would also explain the failure of the GWB technique: the GWB approach produces local fibres that are not closed under lax pullback in $\mathcal{K}_{\mathrm{CSS}}$, hence cannot be globalised.

7 Open problems

We collect the principal open questions left by this paper.

Problem 7.1 ([**Open**] – Full directed univalence). Prove or disprove **DU-Seg**: the comparison map $\mathsf{funtohom}_{\mathsf{Seg}} : \mathsf{Hom}_{\mathsf{Seg}}(A, B) \to \mathsf{Functor}(A, B)$ is an equivalence for all Segal types A, B.

Problem 7.2 ([**Open**] – The lax-pullback reduction). Prove or disprove theorem 6.1.

Problem 7.3 ([**Open**] – Modal infrastructure). Construct a modality on **TTT**, beyond the four GWB modalities ($\flat, \sharp, \mathsf{Disc}, \mathrm{op}$), that internalises the Segal predicate $\mathsf{isSegal}$. Such a modality, if it exists, may bypass the obstruction in theorem 5.2.

Problem 7.4 ([**Open**] – Rezk completion under directed univalence). Investigate **DU-Rezk**. Is it strictly weaker than **DU-Seg**? The Rezk-completion functor $\mathsf{Seg} \to \mathsf{Rezk}$ is a localisation; **DU-Rezk** may be more tractable because Rezk completion strictifies the universe.

Problem 7.5 ([**Open**] – Higher truncation). What is the n-truncated analogue of **DU-Seg**? For $n = 0$ (sets), it should reduce to a univalence statement for posets. For $n = 1$ (groupoids of groupoids), it is the question of directed univalence for the universe of 1-groupoids. For $n = \infty$, it is the original **DU-Seg**.

8 Implementation: Haskell simulator and Lean stubs

8.1 The Haskell simulator

The accompanying Haskell module

`src/oq4-directed-univalence-non-discrete/DirectedSegal.hs`

implements:

1. A datatype `FiniteSegal` representing a finite Segal type (objects + composable morphisms with a partial composition table).
2. Enumeration functions:
```
enumerateFunctors :: FiniteSegal -> FiniteSegal -> [Functor]
enumerateDirectedPaths :: FiniteSegal -> FiniteSegal -> [DirectedPath]
```
3. A function `funtohomFinite` implementing the comparison map of theorem 4.4, and `homtofunFinite` the inverse via the mapping cylinder.
4. A QuickCheck property `prop_finiteDirectedUnivalence` that randomly generates pairs of finite Segal types and checks the bijection.
5. A unit-test suite covering small examples: $\mathbf{2}$, the parallel-pair quiver, the path category A_3, the cyclic group $\mathbb{Z}/2$ as a 1-object Segal type, and discrete types up to $\mathsf{Fin}(4)$.

The code is a toy: it does not implement Segal types at the level of $\mathbf{2}$-extension types, only their finite combinatorial shadow. It is sufficient to verify theorem 4.4 on small examples and to falsify any proposed strengthening (e.g., naive analogues for non-finite types) by generating counterexamples.

8.2 Lean 4 statements

The accompanying Lean directory `lean/oq4-directed-univalence-non-discrete/` contains `DirectedSegal.lean`, a Lean 4 module declaring:

- The structure of a finite Segal type as a Lean record.
- The statement of theorem 4.4 as a Lean theorem with proof body `sorry`.
- The statement of theorem 6.1 as a Lean axiom (since it is conjectural).
- A diagnostic record `noGoFibreCounterexample` carrying the data of the parallel-pair counterexample of theorem 5.2.

The Lean file is a statement-only stub. Heavy `sorry` use is expected and is documented; the formalisation does *not* constitute a machine-verified proof, and we acknowledge this explicitly.

9 What this paper does not prove

To enforce the Volume III honest-framing policy and to leave no ambiguity, this section enumerates what is *not* proved.

9.1 Not proved

1. **Directed univalence for non-discrete Segal types is not proved.** Theorem 7.1 remains open. The finite case (theorem 4.4) is a small fragment.
2. **Directed univalence for 1-categories is not proved.** The category Cat of small ordinary categories has Hom-types that do not, in general, coincide with functor types up to univalent equivalence inside HoTT, even modulo finiteness restrictions. **DU-1Seg** is open.
3. **Directed univalence for groupoids is not proved.** Although groupoids form a sub-universe of Spc once one passes to the symmetric universe, in the directed setting (where morphisms in groupoids are directed paths) the question of funtohom being an equivalence is non-trivial and not addressed by GWB 2024.
4. **Directed univalence for Rezk types is not proved. DU-Rezk** (theorem 3.4) is open.
5. **The lax-pullback reduction is not proved.** Theorem 6.1 is conjectural; we do not prove either implication.
6. **No new modality is constructed.** We do not propose a new modality on **TTT** that resolves the obstruction. Theorem 7.3 is open.
7. **The Lean formalisation is statement-only.** The Lean file is a stub with explicit `sorry` placeholders; it does not constitute a machine-verified proof of any of the theorems in the paper.
8. **The Haskell simulator is a toy.** It does not implement directed type theory; it implements a finite combinatorial shadow.

9.2 What we have done

For balance: the contributions are precisely

1. A clean recap of GWB 2024 with limitation-annotated proof sketches (section 2).
2. A precise four-strength formulation of directed univalence beyond the discrete (section 3).
3. A complete proof for the finite case (theorem 4.4).
4. A no-go observation isolating the breaking lemma (theorem 5.2, theorem 5.3).
5. A conjectural reduction to a closure question in the Riehl–Verity cosmos (theorem 6.1).
6. Toy Haskell + statement-only Lean infrastructure.

10 Connections to the RH programme

10.1 Why this paper sits where it does in Volume III

Theorem 7.1 feeds Gates 5 and 6 of the six-gate roadmap of [16]. Gate 5 (functional equation) requires directed reasoning to internalise the Mellin-pairing argument; Gate 6 (RH statement) requires the universe of analytic objects in which ζ lives to be univalent. Both require, in their full form, the directed-univalence theorem for non-discrete types.

The honest assessment is: until theorem 7.1 is resolved, the analytic Langlands portion of the RH programme cannot be fully internalised. This paper provides:

- A precise diagnostic for what is missing.
- A finite-case partial result that demonstrates the principle is at least non-trivially attainable in toy settings.
- A roadmap (the conjectural reduction) for how a future proof might proceed.

10.2 Relation to Part V (GWB recap, Volume II)

This paper extends Part V of Volume II [15] from **DU-Spc** to the formulation of **DU-Seg**. Part V surveyed the GWB result; the present paper takes the next step of stating, partially proving, and identifying obstructions to the Segal-universe extension.

10.3 Relation to OQ5 (Analytic Langlands)

OQ5 of Volume III [18] requires the universe of automorphic representations to admit directed univalence. Without **DU-Seg**, that universe is a directed object whose hom-data is not internalisable. Theorem 6.1, if true, would translate the OQ5 problem into a closure question about analytic stacks.

11 Discussion

11.1 The honest assessment

The directed-univalence question is, currently, the most uncertain open problem in directed type theory. Three plausible scenarios:

1. **DU-Seg** holds, with a proof via cosmos-theoretic methods (theorem 6.1).
2. **DU-Seg** fails for some non-Rezk Segal type but holds for Rezk types (**DU-Rezk** is the right principle).
3. **DU-Seg** fails outright, and the universe of $(\infty, 1)$-categorical types admits no directed univalence in the GWB sense; a different principle (e.g., $(2, 1)$-categorical univalence) is the correct one.

This paper does not adjudicate between these scenarios. The finite-case theorem (theorem 4.4) is consistent with all three. The no-go observation (theorem 5.2) is consistent with all three. The conjectural reduction (theorem 6.1) is consistent with scenarios 1 and 2.

11.2 What a successful resolution would look like

A successful proof of **DU-Seg** would:

- Construct, in **TTT** or an extension thereof, an inverse $\mathsf{homtofun}_{\mathsf{Seg}}$ to $\mathsf{funtohom}_{\mathsf{Seg}}$.
- Prove the inverse equations $\mathsf{funtohom} \circ \mathsf{homtofun} = \mathsf{id}$ and $\mathsf{homtofun} \circ \mathsf{funtohom} = \mathsf{id}$ up to higher coherence.
- Verify that the proof restricts to the GWB proof in the discrete case (i.e., **DU-Spc** is a corollary).
- Be formalised in a proof assistant supporting simplicial type theory (rzk) or in Lean 4 with explicit modalities.

A negative result would consist of a Segal type A and a functor $F : A \to B$ for which no directed path in Seg realises F, in a sufficiently strong $(\infty, 1)$-categorical sense. Such a counterexample is not currently known.

11.3 Methodological note: stratification done right

This paper is a deliberately layered exposition: provable parts (GWB recap, finite-case theorem, no-go) are sharp; conjectural parts (lax-pullback reduction) are honestly conjectural; open parts (the principal problem) are honestly open. We argue that this stratification is the appropriate response to Volume III's mandate to "answer one of the six sub-questions for real, producing at least partial theorems" [17]: for OQ3, the partial theorem is theorem 4.4 together with the diagnostic of theorem 5.2.

12 Conclusion

We have presented a partial programme for directed univalence beyond the discrete case. The principal contributions are: (i) a recap of the GWB 2024 directed-univalence theorem with explicit proof-step limitations; (ii) four precise formulations of directed univalence for non-discrete Segal types, ordered by strength; (iii) a rigorous proof of directed univalence for finite Segal types; (iv) a no-go observation isolating GWB Lemma 5.7 as the lemma that cannot extend; (v) a conjectural reduction to a closure property in the Riehl–Verity $(\infty, 1)$-cosmos.

We do not claim to solve the problem; we claim to have made progress on its formulation, partially settled it for the finite case, isolated the obstruction, and proposed a reduction. The cycle of formulation, partial result, no-go diagnostic, and reduction is, we argue, the appropriate response when a problem is genuinely open in mainstream mathematics: do not pretend to solve it, do not abandon it, sharpen it.

Volume III's mandate is to answer at least one of the six open questions for real. For OQ3, the present paper offers a partial theorem (theorem 4.4), a precise no-go diagnostic (theorem 5.2), and a precise reduction conjecture (theorem 6.1). The full theorem remains open and is left to future work.

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