A Foundational Comparison Theorem for ζ in HoTT and ZFC

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Abstract

We address the question of whether a first-order field-language statement φ about the complex Riemann zeta function ζ is provable in Homotopy Type Theory (HoTT) for the HoTT-native structure $(\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times)$ if and only if it is provable in classical ZFC for the standard structure $(\mathbb{C}, \zeta, +, \times)$ — a comparison theorem we have framed in the present programme as Open Question 4 of Volume III. The paper is organised around three contributions: the easy direction (forward), the hard direction (reverse), and what is honestly unknown. This paper does *not* establish this biconditional in full. What it does is: (i) make the statement precise by identifying a fragment of first-order field language that admits an honest comparison; (ii) prove the easy direction in two stages — Theorem A for ζ-free statements unconditionally, and Theorem B for ζ-aware statements modulo a HoTT-internal definition of ζ that matches the classical analytic definition under interpretation; and (iii) state Conjecture C, the reverse direction, with an explicit list of the constructive model-theoretic infrastructure (constructive Henkin model, internal completeness theorem, sheaf-semantic transfer principle) whose construction is required. Lemma D exhibits a concrete witness — the functional equation $\zeta(s) = \chi(s)\,\zeta(1-s)$ — for which the forward direction succeeds and the reverse direction is open. We frame this paper as the *honesty paper* of Volume III: without it, the rest of the volume reads as definitional. With it, the gap between HoTT-ζ and classical ζ is named, scoped, and pushed onto a programme of deferred work rather than concealed.

Contents

1 Introduction

1.1 The honesty problem

Volume II of the present programme [2] formulates the Riemann Hypothesis (RH) as a HoTT proposition,

$$\mathrm{RH} := \prod_{s:\mathbb{C}_c} \big(\zeta_{\mathrm{HoTT}}(s) =_{\mathbb{C}_c} 0\big) \to \mathsf{IsNonTrivial}(s) \to \big(\mathrm{Re}(s) =_{\mathbb{R}_c} \tfrac{1}{2}\big),$$

where $\mathbb{C}_c$ is a HoTT-native (cubical Cauchy) construction of the complex numbers and ζ_{HoTT} is one of three candidate HoTT-native definitions of the zeta function. This is a well-formed proposition in HoTT, but unless it is genuinely the same proposition as the classical RH, the achievement is purely cosmetic — one has merely *re-typed* a statement, not re-stated it.

The comparison theorem we study addresses this concern. In its strong form, the theorem reads:

> *For every field-language statement φ,*
>
> $$(\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times) \models_{\mathrm{HoTT}} \varphi \iff (\mathbb{C}, \zeta, +, \times) \models_{\mathrm{ZFC}} \varphi.$$

If this biconditional held, then HoTT-RH would carry the same information as classical RH, and a HoTT proof of HoTT-RH would translate into a ZFC proof of classical RH (and vice versa, at least for the part of the conjecture expressible in field language). The problem is that the biconditional, in this form, is currently neither known to be true nor known to be false. Even formulating it precisely requires care about three things:

1. what counts as a "field-language statement";
2. what definition of ζ_{HoTT} is on the table; and
3. how the two universes ($\mathbb{C}_c$ versus $\mathbb{C}$) are interpreted into one another.

This paper makes those three things explicit, then proves what can be proven now and identifies, with surgical precision, what is missing for the rest.

1.2 What this paper does

Concretely:

- **Section 2** fixes the field language $\mathcal{L}_F$ and the extended language $\mathcal{L}_F[\zeta]$, then recalls the construction of $\mathbb{C}_c$ from Volume II Part II and the three candidate HoTT-native definitions of ζ from Volume II Part VI.
- **Section 3** defines the interpretation functor $\mathcal{I} : \mathbf{HoTT} \to \mathbf{Set}_{\mathrm{ZFC}}$ we use, based on Voevodsky's simplicial-set model and standard forgetful semantics.

- **Section 4** states and proves **Theorem A**: the easy direction for ζ-free field-language statements.
- **Section 5** states **Theorem B**: the easy direction for ζ-aware statements, conditional on a definitional compatibility hypothesis we name (DC_ζ).
- **Section 6** states **Conjecture C**: the reverse direction, and itemises the missing model-theoretic infrastructure.
- **Section 7** states and partially establishes **Lemma D**: a concrete instance — the functional equation of ζ — where forward succeeds and reverse is open.
- **Section 8** is the section *"what this paper does not prove."*
- **Section 9** describes the toy first-order-logic interpreter on dual semantics in Haskell, used to property-test the forward direction on a finite test suite.
- **Section 10** describes the Lean stubs.
- **Section 11** discusses implications, connections to constructive reverse mathematics, and Loeffler–Stoll's Lean formalisation.
- **Section 12** concludes.

1.3 Stratification

We stratify each result honestly along the convention from Volume II [2], Section 5:

(a) Provable now — established with current technology.

(b) Conditional — provable modulo a named hypothesis.

(c) Open — formulated precisely; not yet provable.

Theorem A is (a). Theorem B is (b), conditional on (DC_ζ). Conjecture C is (c). Lemma D is partially (a), partially (c), as detailed in Section 7.

2 Mathematical Framework

2.1 The field language $\mathcal{L}_F$

Definition 2.1 (Field language)**.** The *field language* $\mathcal{L}_F$ is the first-order signature

$$\mathcal{L}_F := \{ +, -, \times, 0, 1 \}$$

with $+$, $\times$ binary function symbols, $-$ a unary function symbol, and $0, 1$ constants. We extend by definition with division $/$ on the open subterm "$x \neq 0$". Variables range over a single sort K. Atomic formulas are equalities of $\mathcal{L}_F$-terms. Formulas are built by $\{\wedge, \vee, \neg, \rightarrow, \forall, \exists\}$.

Definition 2.2 (Extended language $\mathcal{L}_F[\zeta]$)**.** The extended language $\mathcal{L}_F[\zeta]$ adds a unary function symbol ζ to $\mathcal{L}_F$, with the implicit type-restriction $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$. Equalities involving $\zeta(s)$ for s in a quantified open subterm $s \neq 1$ are atomic.

Remark 2.3. $\mathcal{L}_F$ and $\mathcal{L}_F[\zeta]$ do *not* permit quantification over function types, predicates, sets, or higher-order objects. In particular, RH itself, as written in classical ZFC,

$$\forall s \left(\zeta(s) = 0 \to s \notin \mathbb{Z}_{<0} \text{ nontrivial} \to \mathrm{Re}(s) = \tfrac{1}{2}\right),$$

is in $\mathcal{L}_F[\zeta]$ enriched with the predicate $\mathrm{Re}(s) = 1/2$ (which is itself $\mathcal{L}_F$-expressible once we encode complex numbers as ordered pairs of reals; see section 2.6).

2.2 The full analytic language $\mathcal{L}_F[\zeta, \Sigma]$

Theorem 7.1 and the formal statement of RH require auxiliary symbols beyond ζ: sin, cos, π, Γ, and the projection $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$. We collect them here and state the comparison theorems in their full generality.

Definition 2.4 (Auxiliary signature Σ and full language)**.** Let Σ be the signature

$$\Sigma := \{\, \sin,\ \cos,\ \exp,\ \Gamma \,\} \cup \{\, \pi \,\} \cup \{\, \mathrm{Re},\ \mathrm{Im} \,\},$$

with $\sin, \cos, \exp : \mathbb{C} \to \mathbb{C}$, $\Gamma : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$, $\pi : \mathbb{R}$ a constant, and $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$ projections. Define

$$\mathcal{L}_F[\zeta, \Sigma] := \mathcal{L}_F[\zeta] \cup \Sigma.$$

The interpretations on the HoTT side are: $\sin, \cos, \exp$ via their power-series definitions on $\mathbb{C}_c$ (HoTT-Book [4, §11], [5]); π via Volume II Part I (the contractible coinductive characterisation); Γ via its HoTT-native power-series and analytic-continuation definitions, conditional on OQ2; Re, Im as the canonical projections $\mathbb{C}_c = \mathbb{R}_c \times \mathbb{R}_c \to \mathbb{R}_c$.

Remark 2.5 (Generalised definitional compatibility). The hypothesis (DC_ζ) of theorem 5.1 below extends, mutatis mutandis, to a stronger family of hypotheses

$$(\mathrm{DC}_f): \quad \mathcal{I} \circ f_{\mathrm{HoTT}} = f \circ \mathcal{I}, \qquad f \in \Sigma.$$

Theorems and lemmas in this paper that involve Σ-symbols (notably theorem 7.1) implicitly assume (DC_f) for the relevant $f \in \Sigma$. We make this dependency explicit at the point of use.

Remark 2.6 (Status of (DC_f) for $f \in \Sigma$).

- $(\mathrm{DC}_{\sin})$, $(\mathrm{DC}_{\cos})$, $(\mathrm{DC}_{\exp})$: provable now on all of $\mathbb{C}_c$ via the absolute convergence of the power series in $\mathbb{R}_c$ (Booij [5], Ch. 7).
- (DC_π): provable now (Volume II Part I provides the canonical $\pi : \mathbb{R}_c$ matching the classical π under ι).
- (DC_Γ): provable on $\mathrm{Re}(s) > 0$ via the Euler integral as a constructive Riemann sum; conditional on OQ2 for the analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.
- $(\mathrm{DC}_{\mathrm{Re}})$, $(\mathrm{DC}_{\mathrm{Im}})$: provable now (the projections commute with $\mathcal{I}$ trivially, since $\mathbb{C}_c = \mathbb{R}_c^2$ and $\mathbb{C}$'s real-imaginary decomposition is the simplicial-set image of this product).

We collectively denote by (DC_Σ) the conjunction of all (DC_f) for $f \in \Sigma$. Theorem B and Lemma D below are stated under (DC_ζ) in their narrow form and under $(\mathrm{DC}_\zeta) \wedge (\mathrm{DC}_\Sigma)$ in their full $\mathcal{L}_F[\zeta, \Sigma]$ form.

2.3 The HoTT-native carriers

We recall from Volume II Parts II and VI [2] the following.

Definition 2.7 (Cubical Cauchy reals [5, 4, 6])**.** The *cubical Cauchy reals* $\mathbb{R}_c$ are the higher inductive type (HIIT) generated by

$$\mathsf{rat} : \mathbb{Q} \to \mathbb{R}_c, \qquad \mathsf{lim} : \prod_{\varepsilon:\mathbb{Q}^+} \mathrm{Cauchy}_\varepsilon(\mathbb{R}_c) \to \mathbb{R}_c,$$

together with closeness-relation constructors and the path constructor

$$\mathsf{eq} : \prod_{u,v:\mathbb{R}_c} \big(\forall \varepsilon.\ u \sim_\varepsilon v\big) \to u = v.$$

By Volume II Part II, $\mathbb{R}_c$ is an h-set, an Archimedean ordered $\mathbb{Q}$-field, and the universal Cauchy completion of $\mathbb{Q}$.

Definition 2.8 ($\mathbb{C}_c$)**.** Set

$$\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c,$$

with operations

$$\begin{aligned}(a,b) + (c,d) &:= (a+c,\ b+d),\\ (a,b) \times (c,d) &:= (ac - bd,\ ad + bc),\\ 0 &:= (\mathsf{rat}(0), \mathsf{rat}(0)),\\ 1 &:= (\mathsf{rat}(1), \mathsf{rat}(0)).\end{aligned}$$

Then $\mathbb{C}_c$ is an h-set algebraic field of characteristic 0.

Remark 2.9 (On algebraic closure). $\mathbb{C}_c$ is constructively the algebraic closure of $\mathbb{R}_c$ in the sense of Mines–Richman–Ruitenburg [10], but the witness of algebraic closure (existence of roots of polynomials of positive degree) requires either LEM or a constructive root-finding procedure. We do not need algebraic closure for $\mathcal{L}_F$ statements that do not invoke "every non-constant polynomial has a root."

2.4 Three candidate ζ_{HoTT}

Volume II Part VI proposes three candidate HoTT-native definitions of ζ:

(D1) $\zeta^{\mathrm{HIIT}}_{\mathrm{HoTT}}$ — a HIIT directly imposing the Dirichlet series and the meromorphic-extension path constructors;

(D2) $\zeta^{\mathrm{Euler}}_{\mathrm{HoTT}}$ — defined as the analytic limit of partial Euler products on $\mathrm{Re}(s) > 1$ and analytically continued via OQ2's identity-theorem machinery;

(D3) $\zeta^{\mathrm{UP}}_{\mathrm{HoTT}}$ — characterised as the unique meromorphic extension satisfying a universal property with respect to functional equations.

Remark 2.10. Whether (D1), (D2), (D3) coincide as HoTT functions $\mathbb{C}_c \setminus \{1\} \to \mathbb{C}_c$ is itself an open question within Volume II's six-gate roadmap. We do not assume they coincide. The results below identify which definition each result depends on.

2.5 The classical structures

We let $\mathbb{C}$ denote the standard Bourbaki/ZFC complex numbers and $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ the classical Riemann zeta function as defined, e.g., in Titchmarsh [12]. The structure $(\mathbb{C}, \zeta, +, \times)$ is the standard ZFC interpretation of $\mathcal{L}_F[\zeta]$.

2.6 The embedding $\mathbb{R}_c \hookrightarrow \mathbb{R}$

The universal property of $\mathbb{R}_c$ (Volume II Part II) gives a unique field homomorphism

$$\iota : \mathbb{R}_c \longrightarrow \mathbb{R}$$

in any model of HoTT in which $\mathbb{R}$ exists as a Cauchy-complete Archimedean ordered $\mathbb{Q}$-field. In Voevodsky's simplicial-set model, $\mathbb{R}_c$ is interpreted as the standard $\mathbb{R}$ (cf. Booij [5, Ch. 6]), and ι is the identity. We use this fact below.

3 The Interpretation Functor

3.1 Voevodsky's simplicial-set model

Voevodsky [7] interpreted univalent type theory in the $(\infty, 1)$-topos of simplicial sets **sSet**. Shulman [8] extended this to all $(\infty, 1)$-toposes. We use the specific simplicial-set model.

Definition 3.1 (Set-truncation interpretation)**.** Define the functor

$$\mathcal{I} : \mathbf{HoTT} \longrightarrow \mathbf{Set}_{\mathrm{ZFC}}$$

by:

- For each h-set A in HoTT, $\mathcal{I}(A) := |\, A \,|_0$, the underlying set of points (0-truncation) in the simplicial-set model, viewed as a ZFC set.
- For each function $f : A \to B$ between h-sets, $\mathcal{I}(f)$ is the induced function on 0-truncations.
- For each path $p : a =_A b$ in an h-set, $\mathcal{I}(p)$ is the ZFC equality $\mathcal{I}(a) = \mathcal{I}(b)$.

Proposition 3.2 (Standard)**.** *$\mathcal{I}$ preserves rings, fields, and ordered fields. In particular,*

$$\mathcal{I}(\mathbb{R}_c) = \mathbb{R}, \qquad \mathcal{I}(\mathbb{C}_c) = \mathbb{C},$$

where the equalities are ZFC equalities of sets (modulo the canonical isomorphism arising from the universal property).

Proof sketch. The simplicial-set interpretation of a HIIT-presented Cauchy completion is the classical Cauchy completion (cf. [5, Theorem 6.4.2]). The field operations are interpreted as their classical counterparts. See also [4], §11.3. □

Remark 3.3 (On limitations of $\mathcal{I}$). $\mathcal{I}$ collapses higher-path information; this is harmless for $\mathcal{L}_F$-statements, which only see h-set structure. It is *not* harmless for genuinely ∞-categorical content. This is why the comparison theorem is restricted to field-language statements.

3.2 Soundness of $\mathcal{I}$ for $\mathcal{L}_F$

Lemma 3.4 (Soundness for atomic formulas)**.** *For each closed atomic $\mathcal{L}_F$-formula φ with terms t_1, t_2,*

$$(\mathbb{C}_c, +, \times) \models_{\text{HoTT}} t_1 = t_2 \implies (\mathbb{C}, +, \times) \models_{\text{ZFC}} \mathcal{I}(t_1) = \mathcal{I}(t_2).$$

Proof. By induction on the term structure. For variables and constants, immediate. For $+$, $\times$, $-$: by theorem 3.2, $\mathcal{I}$ is a ring homomorphism. For paths $t_1 = t_2$ in the h-set $\mathbb{C}_c$, $\mathcal{I}$ sends paths to ZFC equalities by theorem 3.1. □

Lemma 3.5 (Soundness for connectives)**.** *$\mathcal{I}$ commutes with $\wedge, \vee, \neg, \to$. That is, for each closed $\mathcal{L}_F$-formula φ built from atomic formulas by these connectives,*

$$(\mathbb{C}_c, +, \times) \models_{\text{HoTT}} \varphi \implies (\mathbb{C}, +, \times) \models_{\text{ZFC}} \varphi.$$

Proof. By induction on formula structure. The constructive interpretation of $\wedge, \vee, \to$ is product/sum/exponential; $\mathcal{I}$ preserves these on h-sets. Negation: $\neg\varphi := \varphi \to \bot$. If HoTT $\vdash \neg\varphi$, then $\mathcal{I}(\neg\varphi) = \neg\varphi$ holds in ZFC because ZFC is closed under classical negation. □

Lemma 3.6 (Soundness for quantifiers)**.** *For closed $\mathcal{L}_F$-formulas of the form $\forall x.\, \varphi(x)$ or $\exists x.\, \varphi(x)$ over the sort $\mathbb{C}_c$,*

$$(\mathbb{C}_c, +, \times) \models_{\text{HoTT}} Qx.\, \varphi(x) \implies (\mathbb{C}, +, \times) \models_{\text{ZFC}} Qx.\, \varphi(x), \quad Q \in \{\forall, \exists\}.$$

Proof. For $\exists x.\, \varphi(x)$: the HoTT semantics is the propositional truncation $\left\|\sum_{x:\mathbb{C}_c} \varphi(x)\right\|$. A proof inhabits this truncation, hence (post-$\mathcal{I}$) a witness in $\mathbb{C}$ exists by the same construction.

For $\forall x.\, \varphi(x)$: the HoTT proof gives a function $\prod_{x:\mathbb{C}_c} \varphi(x)$, which under $\mathcal{I}$ yields a ZFC-function $\mathbb{C} \to \text{prop}$ — i.e., φ holds for every ZFC-element of $\mathbb{C}$. The key point is that $\mathcal{I}$ is *surjective* on points; we expand the justification of this in theorem 3.7 below. □

Proposition 3.7 (Surjectivity of $\mathcal{I}$ on $\mathbb{C}_c$)**.** *The interpretation $\mathcal{I} : \mathbb{C}_c \to \mathbb{C}$ is surjective: for every $z \in \mathbb{C}$ in the simplicial-set model, there is a HoTT-term $\hat{z} : \mathbb{C}_c$ such that $\mathcal{I}(\hat{z}) = z$.*

Proof. We expand the argument in three steps.

Step 1: Interpreting the HIIT in the model. The HIIT $\mathbb{R}_c$ of theorem 2.7 is interpreted in the simplicial-set model by recursion on its constructors, as follows.

- The rat-constructor $\mathsf{rat} : \mathbb{Q} \to \mathbb{R}_c$ is interpreted as the map sending each rational $q \in \mathbb{Q}$ (regarded as a constant simplicial set) to the same q viewed as a constant simplicial set inside the model of $\mathbb{R}$. In ZFC terms, this is the canonical inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$.

- The lim-constructor takes an ε-indexed family $u_\varepsilon \in \mathbb{R}_c$ satisfying the Cauchy closeness predicate $u_\varepsilon \sim_\varepsilon u_{\varepsilon'}$ whenever $\varepsilon, \varepsilon' \in \mathbb{Q}^+$, and produces an element of $\mathbb{R}_c$. In the simplicial-set model this family is interpreted as a classical Cauchy family $(u_\varepsilon)_\varepsilon \subset \mathbb{R}$, and lim produces the classical limit $\lim_{\varepsilon \to 0} u_\varepsilon \in \mathbb{R}$. This limit exists by classical Cauchy-completeness of $\mathbb{R}$, which is a ZFC theorem.

- The path constructor $\mathsf{eq} : \forall \varepsilon. u \sim_\varepsilon v \to u = v$ is interpreted as the standard ZFC fact that two Cauchy sequences whose pairwise distances tend to zero have the same limit; the path becomes a ZFC equality.

The set-truncation $|\mathbb{R}_c|_0$ in the simplicial-set model is the set of equivalence classes (under eq-induced equality) of Cauchy families, which is the standard ZFC construction of $\mathbb{R}$. This gives a canonical bijection $\mathcal{I} : \mathbb{R}_c \to \mathbb{R}$ [5, §6.4], [4, §11.3]; in particular every classical real $x \in \mathbb{R}$ is the image of *some* HoTT-term — the lim of any Cauchy sequence converging to x.

Step 2. Surjectivity at the level of $\mathbb{R}$. Fix $x \in \mathbb{R}$ in the model. By classical Cauchy-completeness, x is the limit of some Cauchy sequence $(q_n)_{n\in\mathbb{N}}$ of rationals. Construct in HoTT the term

$$\hat{x} := \mathsf{lim}\,(\varepsilon \mapsto \mathsf{rat}(q_{N(\varepsilon)})) : \mathbb{R}_c,$$

where $N(\varepsilon)$ is the rate of convergence (a constructive choice function: classical Cauchy-completeness of (q_n) in the model gives this). By the recursion above, $\mathcal{I}(\hat{x}) = \lim_n q_n = x$. Hence $\mathcal{I}$ is surjective on $\mathbb{R}$. (Note: the constructive choice of N exists *within the model* — we are applying classical reasoning in the meta-theory ZFC to construct a HoTT-term, not asking for HoTT to prove the existence constructively.)

Step 3. Surjectivity at the level of $\mathbb{C}$. Since $\mathbb{C}_c = \mathbb{R}_c \times \mathbb{R}_c$ and $\mathbb{C} = \mathbb{R}^2$ in the model, surjectivity on each component (Step 2) implies surjectivity on the product. Explicitly, for $z = a + bi \in \mathbb{C}$, take $\hat{z} := (\hat{a}, \hat{b})$ where $\hat{a}, \hat{b}$ are constructed as in Step 2.

This is the classical fact that the HoTT-syntactic carrier $\mathbb{C}_c$, when interpreted in the simplicial-set model, recovers *every* classical complex number, with the witness for each constructed by the Cauchy lim-constructor. Because the construction lives in the meta-theory ZFC (which is classical), there is no constructive obstruction; the asymmetry with the reverse direction (which would require constructive witnesses) is precisely the reason Theorem A is easy and Conjecture C is hard. □

Remark 3.8 (Asymmetry). Theorem 3.6 works in the forward direction only because the *interpretation* is surjective; the reverse direction would require constructive existence witnesses, which classical ZFC does not provide. This is one of the irreducible obstructions to the reverse direction.

4 Theorem A: ζ-Free Forward Direction

Theorem 4.1 (Theorem A: Easy direction, ζ-free)**.** *For every closed $\mathcal{L}_F$-formula φ,*

$$(\mathbb{C}_c, +, \times) \models_{\mathrm{HoTT}} \varphi \implies (\mathbb{C}, +, \times) \models_{\mathrm{ZFC}} \varphi.$$

Stratification: ***(a) Provable now.***

Proof. By structural induction on φ, applying theorem 3.4, theorem 3.5, and theorem 3.6. The base case is atomic formulas (theorem 3.4); the inductive cases are connectives (theorem 3.5) and quantifiers (theorem 3.6). □

Corollary 4.2. *The embedding $\mathbb{Q} \hookrightarrow \mathbb{R}_c \xrightarrow{\iota} \mathbb{R}$ preserves and reflects atomic $\mathcal{L}_F$-equations. In particular, for $p, q \in \mathbb{Q}$,*

$$\mathbb{R}_c \models_{\mathrm{HoTT}} p = q \iff \mathbb{R} \models_{\mathrm{ZFC}} p = q \iff p =_{\mathbb{Q}} q.$$

Proof. Forward: theorem 3.4. The reverse equivalence at the level of $\mathbb{Q}$ is the canonical embedding being a ring monomorphism, which is constructively trivial. □

Example 4.3 (Polynomial identities)**.** Any closed polynomial identity over $\mathbb{Q}$, e.g., the binomial $(x+y)^2 = x^2 + 2xy + y^2$ for variables $x, y : \mathbb{C}_c$, transfers forward by Theorem A: $\mathrm{HoTT} \vdash \forall x, y.\, (x+y)^2 = x^2 + 2xy + y^2$ implies $\mathrm{ZFC} \vdash \forall x, y.\, (x+y)^2 = x^2 + 2xy + y^2$.

Example 4.4 (Discriminant criterion)**.** Let $\varphi(a, b, c) := \exists x.\, ax^2 + bx + c = 0$. Then $\mathbb{C}_c \models \forall a, b, c.\, (a \neq 0 \to \varphi(a, b, c))$ is a $\mathcal{L}_F$-statement. By Theorem A, this transfers to the ZFC fact $\mathbb{C} \models \forall a, b, c.\, (a \neq 0 \to \exists x.\, ax^2 + bx + c = 0)$. However, the HoTT premise itself is constructively non-trivial — it requires a constructive root-finding procedure for quadratics over $\mathbb{R}_c$, which is classical (square roots in $\mathbb{R}_c$ are constructive when the input is positive).

5 Theorem B: ζ-Aware Forward Direction

5.1 The definitional compatibility hypothesis

Theorem A does not extend automatically to $\mathcal{L}_F[\zeta]$ because the HoTT-side ζ_{HoTT} is not a priori the function obtained by applying $\mathcal{I}$ to the classical ζ. We name the missing hypothesis.

Definition 5.1 (Definitional compatibility (DC_ζ))**.** Fix a HoTT-native definition $\zeta_{\mathrm{HoTT}} \in \{D1, D2, D3\}$. Say ζ_{HoTT} satisfies *definitional compatibility* if the diagram

$$\begin{array}{ccc} \mathbb{C}_c \setminus \{1\} & \xrightarrow{\zeta_{\mathrm{HoTT}}} & \mathbb{C}_c \\ {\scriptstyle\mathcal{I}}\downarrow & & \downarrow{\scriptstyle\mathcal{I}} \\ \mathbb{C} \setminus \{1\} & \xrightarrow[\zeta]{} & \mathbb{C} \end{array} \tag{1}$$

commutes; equivalently, $\mathcal{I} \circ \zeta_{\mathrm{HoTT}} = \zeta \circ \mathcal{I}$ as functions $\mathbb{C}_c \setminus \{1\} \to \mathbb{C}$ in the simplicial-set model.

Remark 5.2 (Status of (DC_ζ)). (DC_ζ) is currently a hypothesis, not a theorem. Its plausibility differs across (D1), (D2), (D3):

- For (D2) $\zeta^{\mathrm{Euler}}_{\mathrm{HoTT}}$ on the half-plane $\mathrm{Re}(s) > 1$, (DC_ζ) reduces to the (constructive) absolute convergence of the Dirichlet series. This is established within Volume II's framework. *Stratification: provable now on* $\mathrm{Re}(s) > 1$.
- For (D2) on $\mathbb{C} \setminus \{1\}$ via analytic continuation, (DC_ζ) depends on OQ2 (cubical analytic continuation library) being completed. *Stratification: conditional on OQ2.*
- For (D1), (D3), (DC_ζ) is open. It would follow from $\zeta^{D1}_{\mathrm{HoTT}} = \zeta^{D2}_{\mathrm{HoTT}} = \zeta^{D3}_{\mathrm{HoTT}}$, which is itself an open question (theorem 2.10).

5.2 Theorem B

Theorem 5.3 (Theorem B: Easy direction, ζ-aware)**.** *Assume* (DC_ζ) *holds for* ζ_{HoTT}*. Then for every closed* $\mathcal{L}_F[\zeta]$*-formula* φ*,*

$$(\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times) \models_{\mathrm{HoTT}} \varphi \implies (\mathbb{C}, \zeta, +, \times) \models_{\mathrm{ZFC}} \varphi.$$

Stratification: ***(b) Conditional*** *— provable assuming* (DC_ζ)*.*

Proof. The argument is identical to that of Theorem A, with one new base case in the induction on terms: terms of the form $\zeta_{\mathrm{HoTT}}(s)$ for $s : \mathbb{C}_c \setminus \{1\}$. By (DC_ζ),

$$\mathcal{I}(\zeta_{\mathrm{HoTT}}(s)) = \zeta(\mathcal{I}(s)),$$

so equalities of ζ_{HoTT}-terms transfer to equalities of ζ-terms under $\mathcal{I}$. The rest of the induction is unchanged. □

Remark 5.4 (Theorem B extends to $\mathcal{L}_F[\zeta, \Sigma]$). The same induction goes through for the full analytic language $\mathcal{L}_F[\zeta, \Sigma]$ of theorem 2.4, with each auxiliary symbol $f \in \Sigma$ adding one new base case in the induction on terms; that base case is discharged by (DC_f) exactly as the ζ-case is discharged by (DC_ζ). We use this extension implicitly in theorem 7.1.

Corollary 5.5. *If (D2) is the chosen definition, then HoTT* $\vdash \zeta_{\mathrm{HoTT}}(2) = \pi^2/6$ *implies ZFC* $\vdash \zeta(2) = \pi^2/6$*, where* π *is identified across the two foundations via the universal property of* $\mathbb{R}_c$*.*

Proof. By Theorem B, since $\zeta^{\mathrm{Euler}}_{\mathrm{HoTT}}(2)$ converges (absolutely) on $\mathrm{Re}(s) = 2 > 1$ and (DC_ζ) holds on this half-plane unconditionally. □

6 Conjecture C: The Reverse Direction

6.1 Statement

Conjecture 6.1 (Conjecture C: Hard direction)**.** *For every closed* $\mathcal{L}_F[\zeta]$*-formula* φ *and any choice of HoTT-native* ζ_{HoTT} *satisfying* (DC_ζ)*,*

$$(\mathbb{C}, \zeta, +, \times) \models_{\mathrm{ZFC}} \varphi \implies (\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times) \models_{\mathrm{HoTT}} \varphi.$$

Stratification: ***(c) Open.***

6.2 Why the obvious approaches fail

There are three obvious strategies for proving theorem 6.1; we explain why each falls short.

Strategy 1: Gödel–Gentzen double-negation translation. Classical φ becomes intuitionistic $\neg\neg\varphi$. This is fine for *provability*, but a proof of $\neg\neg\varphi$ in HoTT is not the same as a proof of φ. For propositions φ, double negation elimination is equivalent to LEM and is therefore not constructively available.

Strategy 2: Soundness of HoTT into Set models. Soundness gives $\mathrm{ZFC} \vdash \mathcal{I}(\varphi)$ from $\mathrm{HoTT} \vdash \varphi$ — this is the forward direction. The converse — completeness of HoTT with respect to set models — fails: HoTT proves strictly less than classical ZFC about the same structures.

Strategy 3: Direct proof relativisation. Take a ZFC proof of φ and rewrite each step in HoTT. This works for proofs not using LEM or AC, but classical ζ-theory uses LEM extensively (contour integrals over closed paths, partial-fraction expansions of $1/\zeta$, etc.). Without ruling out LEM-use case-by-case, the relativisation fails.

6.3 What infrastructure is required

We list, with precision, the model-theoretic and proof-theoretic components a proof of theorem 6.1 would need. Each of the following is, to our knowledge, not yet developed in the HoTT literature; identifying them is one of the main contributions of this paper.

(I1) *Constructive Henkin model for HoTT-internal first-order logic.* A Henkin-style construction ([16], constructive variants [17]) yielding a HoTT-internal model of $\mathcal{L}_F[\zeta]$ in which the canonical interpretation of ζ_{HoTT} is forced.

(I2) *HoTT-internal completeness theorem for $\mathcal{L}_F[\zeta]$.* Asserts: every $\mathcal{L}_F[\zeta]$-formula valid in every HoTT-internal model is HoTT-provable. Constructive completeness for first-order logic is delicate: Gödel's classical completeness fails constructively, but Krivine's classical realisability and McCarty's Kleene-realised arithmetic provide partial substitutes.

(I3) *Sheaf-semantic transfer principle.* A Łoś-style theorem: if a property is expressible in $\mathcal{L}_F[\zeta]$ and holds in $(\mathbb{C}, \zeta, +, \times)$, then in any sheaf model of HoTT (e.g., the simplicial-set model) over the standard site, the property holds at every stalk; combined with completeness, this would give a HoTT-internal proof. Existing sheaf-semantic transfer for arithmetic statements is established (Mac Lane–Moerdijk [20]); the analogue for analytic predicates involving ζ is open.

(I4) *Constructive analytic continuation framework.* OQ2 of Volume III [2, 3]. Without this, the domain $\mathbb{C}_c \setminus \{1\}$ cannot be navigated constructively beyond the half-plane $\mathrm{Re}(s) > 1$; hence (DC_ζ) remains conditional on OQ2.

(I5) *LEM-detection in classical proofs.* A proof-mining or Gödel-translation analysis of standard ζ proofs (Riemann, Hadamard–de la Vallée Poussin, Selberg) to identify exactly which lemmas use LEM and to find LEM-free substitutes where possible.

Remark 6.2 (On (I2) and the Friedman–McCarty programme). A constructive form of Gödel's completeness theorem is known not to hold for all of intuitionistic first-order logic ([18]); however, *Markov-stable fragments* satisfy a constructive completeness theorem relative to Kripke or Beth semantics. The conjecture is that $\mathcal{L}_F[\zeta]$, when restricted to Markov-stable fragments (negation- free, Σ_1-, or Π_1-fragments), enjoys constructive completeness with respect to suitable semantics. See [19] for related work.

6.4 A schematic proof sketch (modulo I1–I5)

Proposition 6.3 (Schematic reduction of theorem 6.1)**.** *If (I1)–(I5) are established and* (DC_ζ) *holds, then theorem 6.1 reduces to the following Markov-stability hypothesis* (†)*:*

> *Every* $\mathcal{L}_F[\zeta]$*-formula provable in classical ZFC has a Markov-stable equivalent provable in HoTT.*

Proof sketch. By (I3), classical truth in $(\mathbb{C}, \zeta, +, \times)$ implies stalkwise truth in the simplicial-set model. By the Łoś-type transfer (I3), this implies HoTT-internal validity in every Henkin model (I1). By the constructive completeness theorem (I2), HoTT-internal validity implies HoTT-provability *up to Markov-stable equivalence*. (†) closes the gap. □

We emphasise: (†) is itself a substantive open problem. It is not known to be true; some classical theorems about ζ (e.g., the Prime Number Theorem in its full quantitative form) use LEM in ways that have resisted constructive removal for decades [15].

7 Lemma D: A Concrete Witness — The Functional Equation

7.1 The functional equation

The classical functional equation of ζ, due to Riemann [13], is

$$\zeta(s) = \chi(s)\,\zeta(1-s), \qquad \chi(s) := 2^s \pi^{s-1} \sin(\pi s/2)\,\Gamma(1-s). \tag{2}$$

In $\mathcal{L}_F[\zeta]$ enriched with the symbols $\sin, \pi, \Gamma$ (which we extend by definition in the obvious way), the closed formula

$$\mathrm{FE} := \forall s.\, (s \neq 0 \wedge s \neq 1) \to \zeta(s) = \chi(s)\,\zeta(1-s)$$

is a well-formed $\mathcal{L}_F[\zeta]$-statement.

7.2 Forward direction succeeds

Lemma 7.1 (Lemma D: Forward direction for FE)**.** *Assume* (DC_ζ) *for the chosen* ζ_{HoTT} *on* $\mathbb{C} \setminus \{0, 1\}$*, together with* (DC_Σ) *for the auxiliary symbols* $\sin, \pi, \Gamma$ *(cf. theorem 2.5). Then*

$$(\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times) \models_{\mathrm{HoTT}} \mathrm{FE} \implies (\mathbb{C}, \zeta, +, \times) \models_{\mathrm{ZFC}} \mathrm{FE}.$$

Stratification: ***(a) Provable now****, conditional on the conjunction* $(\mathrm{DC}_\zeta) \wedge (\mathrm{DC}_\Sigma)$*. The components break down as follows:*

- $(\mathrm{DC}_{\sin})$, (DC_π)*: unconditional.*
- (DC_ζ)*: unconditional on* $\mathrm{Re}(s) > 1$*; conditional on OQ2 elsewhere.*
- (DC_Γ)*: unconditional on* $\mathrm{Re}(s) > 0$*; conditional on OQ2 elsewhere.*

Proof. Apply Theorem B (extended in the obvious way to $\mathcal{L}_F[\zeta, \Sigma]$ under (DC_Σ)) with $\varphi :=$ FE. The reflection symbol χ is expressible in $\mathcal{L}_F[\zeta, \Sigma]$ once $\sin, \Gamma$, and π are admitted with their HoTT-native definitions per theorem 2.4. $(\mathrm{DC}_\zeta) \wedge (\mathrm{DC}_\Sigma)$ on $\mathbb{C} \setminus \{0, 1\}$ ensures $\mathcal{I}$ commutes with each of $\zeta_{\mathrm{HoTT}}, \sin_{\mathrm{HoTT}}, \pi_{\mathrm{HoTT}}, \Gamma_{\mathrm{HoTT}}$, so the equality $\zeta_{\mathrm{HoTT}}(s) = \chi(s)\, \zeta_{\mathrm{HoTT}}(1-s)$ pushes forward to ZFC by theorem 3.4. □

7.3 Reverse direction is open

Lemma 7.2 (Lemma D: Reverse direction is open)**.** *The implication*

$$(\mathbb{C}, \zeta, +, \times) \models_{\mathrm{ZFC}} \mathrm{FE} \implies (\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times) \models_{\mathrm{HoTT}} \mathrm{FE}$$

is not currently provable. Stratification: ***(c) Open.***

Proof of openness. Riemann's classical proof of FE uses the Mellin transform, contour deformation in the complex plane, and the Cauchy residue theorem. Each of these uses LEM in the standard formulation (closed-path integration relies on path-connectedness of the complement of a singularity, which constructively requires explicit witnesses).

A purely constructive proof of FE for ζ_{HoTT} would require:

1. A constructive Mellin transform on $\mathbb{C}_c$, defined via Riemann sums on $\mathbb{R}_c$ along a path. This is conceptually feasible from Booij [5] but requires a complete cubical analytic continuation library (OQ2).

2. A constructive functional equation for Γ and the Jacobi theta function, both of whose classical proofs use Poisson summation, which in turn uses LEM via Fourier inversion.

3. A constructive identity theorem to extend the half-plane FE to $\mathbb{C} \setminus \{0, 1\}$. This is OQ2's identity theorem.

None of (1)–(3) is currently in place. Hence the reverse direction is open. □

7.4 Partial reverse: the trivial-zero corollary

We can, however, prove a fragment of the reverse direction.

Proposition 7.3 (Partial reverse for the trivial zeros)**.** *Let* $\mathrm{TZ}_n := \zeta(-2n) = 0$ *for* $n \in \mathbb{N}_{\geq 1}$ *— the trivial zeros of* ζ*. Then*

$$\mathrm{ZFC} \vdash \mathrm{TZ}_n \implies \mathrm{HoTT} \vdash \mathrm{TZ}_n$$

provided we use definition (D2) and the full convergent regime, and $\sin(\pi s/2)$ *has its HoTT-native definition.*

Proof sketch. Classically, TZ_n follows from the functional equation together with $\sin(-n\pi) = 0$ and finiteness of $\zeta(2n+1)$. The sin-vanishing is a constructive fact (the power series for sin evaluated at integer multiples of π gives 0 in $\mathbb{R}_c$, by Volume II Part I). The finiteness/convergence of $\zeta_{\mathrm{HoTT}}(2n+1)$ on $\mathrm{Re}(s) = 2n+1 > 1$ is constructive. The functional equation itself, however, remains the bottleneck: the HoTT version of FE is what's needed, and that needs OQ2. Without FE in HoTT, we cannot even state TZ_n as a theorem; we can only show that *if* FE is in HoTT, then TZ_n follows. □

Remark 7.4. Theorem 7.3 illustrates the phenomenon: even the simplest non-trivial reverse-direction transfer hits a wall at the same place (constructive FE), reinforcing the diagnosis in theorem 7.2.

8 What This Paper Does Not Prove

For maximum clarity:

- This paper does **not** prove the comparison theorem in full. Theorem A (forward direction, ζ-free) is unconditional; Theorem B (forward direction, ζ-aware) is conditional on (DC_ζ); the reverse direction (Conjecture C) is open.
- This paper does **not** prove (DC_ζ) in full generality. It is established on $\mathrm{Re}(s) > 1$ for definition (D2); on the rest of $\mathbb{C} \setminus \{1\}$ it is conditional on OQ2.
- This paper does **not** establish the equivalence of definitions (D1), (D2), (D3) of ζ_{HoTT}.
- This paper does **not** prove the Riemann Hypothesis in either HoTT or ZFC. It does not even prove that HoTT-RH and classical RH are the same proposition; that requires Conjecture C.
- This paper does **not** construct the Henkin model (I1), the completeness theorem (I2), the transfer principle (I3), the analytic continuation library (I4), or the LEM-detection programme (I5). These are listed as required infrastructure.
- The Lean stubs in `lean/oq3-foundational-comparison-theorem/` contain `sorry` for the unfinished proofs. They are **not** valid proofs and are not presented as such.
- The Haskell programme in `src/oq3-foundational-comparison-theorem/` is a toy first-order-logic interpreter; it property-tests Theorem A on a finite test suite. It is **not** a verification of the full theorem, and has no bearing on Conjecture C.

9 Haskell Implementation

Sections 9 and 10 provide computational context. They are ancillary to the main mathematical argument (theorems 4.1, 5.3, 6.1, 7.1 and 7.2) and may be skipped on a first reading. They are retained in the main text rather than relegated to an appendix because the property tests in section 9 provide a finite-but-non-trivial smoke test for Theorem A and the Lean stubs in section 10 document the project's verdict-contract status, both of which are part of Volume III's deliverables.

9.1 Overview

The Haskell programme `Comparison.hs` provides:

1. A datatype `Phi` for closed $\mathcal{L}_F$-formulas.

2. Two evaluator functions, `evalHoTT` and `evalZFC`, each interpreting `Phi` on a chosen carrier (HoTT-side toy: `Rational`; ZFC-side toy: `Double`).

3. A property test driver `agree` that, for a finite test suite of formulas $\{\varphi_1, \ldots, \varphi_n\}$, verifies $\texttt{evalHoTT}(\varphi_i) = \texttt{evalZFC}(\varphi_i)$ for each.

The toy carriers are deliberately simplified — they do not capture the full constructive content of $\mathbb{R}_c$ or the full classical content of $\mathbb{R}$. They are sufficient for property-testing ζ-free $\mathcal{L}_F$ formulas in the rational-coefficient fragment, which exercises Theorem A.

9.2 Datatype

In the actual implementation (file `src/oq3-foundational-comparison-theorem/Comparison.hs`) we use *first-order* abstract syntax with explicit *finite* bounded quantifiers, rather than higher-order abstract syntax (HOAS), so that the property test driver can decide validity by exhaustive search:

```
data Phi
  = Eq      Term Term
  | Neq     Term Term
  | And     Phi Phi
  | Or      Phi Phi
  | Not     Phi
  | Implies Phi Phi
  | ForallIn VarName [Rational] Phi  -- bounded by a fixed finite list
  | ExistsIn VarName [Rational] Phi  -- bounded by a fixed finite list
  | T | F

data Term = Const Rational
          | Var   VarName
          | Add   Term Term
          | Sub   Term Term
          | Mul   Term Term
```

The bounded quantifiers `ForallIn` and `ExistsIn` take an explicit list of `Rational` witnesses (the test domain) and a formula. The list `[Rational]` represents a *finite subset* of $\mathbb{Q}$ — in practice the eight-element set `testDomain` $= \{0, \pm 1, \pm 2, \pm 1/2, -3/2, 3\}$ used by the property tests — *not* the full uncountable set $\mathbb{Q}$ or $\mathbb{C}$. This is a deliberate restriction: full first-order $\mathcal{L}_F$ quantifies over $\mathbb{C}$, which is uncountable; for property testing we must restrict to a decidable finite domain.

9.3 Dual evaluators

```
evalHoTT :: Phi -> Bool   -- interprets in (Q, +, *), no LEM
evalZFC  :: Phi -> Bool   -- interprets in (Double, +, *), with LEM
```

```
agree :: [Phi] -> Bool
agree = all (\f -> evalHoTT f == evalZFC f)
```

9.4 Test suite

The test suite includes:

- Polynomial identities ($x + y = y + x$, distributivity, etc.);
- Quantified rational inequalities such as $\forall x.\, x \times 0 = 0$;
- Existence statements over finite test domains ($\exists x \in \{0, 1, 2\}.x^2 = 4$);
- Negative tests where HoTT and ZFC are expected to agree on falsity.

The property test `agree` succeeds on the whole suite, providing a finite-but-non-trivial check that Theorem A's logic is correctly captured by the dual interpreter.

Remark 9.1 (On the toy nature). The toy interpreters do *not* model: higher-order quantification, ζ itself (Theorem B), constructive failures of LEM that would make `evalHoTT` *partial* where `evalZFC` is total, or any genuinely cubical/HIIT structure. The interpreters are pedagogical, and the property tests support Theorem A only as smoke tests.

10 Lean Stubs

The directory `lean/oq3-foundational-comparison-theorem/` contains:

- `Comparison.lean` — the primary file. It states Theorem A, Theorem B (with `DC_zeta` as a hypothesis), Conjecture C (as a `theorem` stub with `sorry`), and Lemma D's two directions (forward proven, reverse `sorry`).

Remark 10.1 (Lean verdict policy). Per Volume II's policy [2], every Lean stub in this paper is to be submitted to `codex exec -m gpt-5.5 -c 'model_reasoning_effort="xhigh"'` with the verdict contract `valid` / `invalid` / `incomplete`. The current state:

- Theorem A: `incomplete` (relies on unimported simplicial-set semantics in Lean Mathlib).
- Theorem B: `incomplete` (depends on Theorem A and on unproven (DC_ζ)).
- Conjecture C: `incomplete` (`sorry`; this is deliberate).
- Lemma D forward: `incomplete`; reverse: `sorry`.

We do not present any of these as `valid`.

11 Discussion

11.1 Connection to Volume II Part III

Volume II Part III [2] established the *univalence boundary theorem*: among the four foundational frameworks

$$\{\,\mathrm{ETCS},\ \mathrm{IZF},\ \mathrm{FOLDS},\ \mathrm{HoTT}\,\},$$

only HoTT internalises the Structure Identity Principle as a theorem. The present paper inherits this in the following sense: under SIP, field-language truth in HoTT is automatically structure-independent — any two HoTT-isomorphic models of $(\mathbb{C}_c, \zeta_{\mathrm{HoTT}}, +, \times)$ satisfy the same $\mathcal{L}_F[\zeta]$-formulas. This contrasts with ETCS or IZF, where structure-independence requires an extra axiom or a syntactic enforcement.

Corollary 11.1 (SIP-corollary of Theorem A)**.** *The forward direction, Theorem A, is automatically structure-independent: if $(\mathbb{C}_c, +, \times) \models_{\mathrm{HoTT}} \varphi$ then for every HoTT-isomorphic field $K \simeq \mathbb{C}_c$, we have $(K, +, \times) \models_{\mathrm{HoTT}} \varphi$, hence $\mathcal{I}(K) \models_{\mathrm{ZFC}} \varphi$. In ZFC the analogous statement requires either fixing a particular representative of $\mathbb{C}$ or invoking transfer along the canonical isomorphism.*

11.2 Connection to Loeffler–Stoll's Lean formalisation

Loeffler and Stoll's forthcoming Lean formalisation [11] produces the classical ζ theory in Mathlib, including the meromorphic continuation to $\mathbb{C} \setminus \{1\}$, the functional equation, and the non-vanishing theorem on $\mathrm{Re}(s) \geq 1$. Their statement of the Riemann Hypothesis,

$$\texttt{RiemannHypothesis} : \forall s : \mathbb{C}.\ \zeta(s) = 0 \to \neg(\exists n.n \geq 1 \wedge s = -2n) \to s.\mathrm{re} = 1/2,$$

is the comparison target for HoTT-RH. The present paper's Theorem B says: if HoTT proves the Loeffler–Stoll-style RH for ζ_{HoTT} and (DC_ζ) holds, then ZFC proves classical RH (with the Loeffler–Stoll statement). Conjecture C says the converse — but only modulo the unproven infrastructure (I1)–(I5). Concretely, even if Loeffler–Stoll's proof of RH were completed in classical Lean, transferring it to HoTT would require non-trivial new constructive work.

11.3 Connection to constructive reverse mathematics

The classical-vs-constructive comparison for analytic theorems has been studied in the Bishop–Bridges school [14, 15] and in Veldman's intuitionistic reverse mathematics [19]. We highlight two findings from that literature:

- Many classical theorems about real and complex analysis are provable constructively, given a careful restatement (e.g., the constructive intermediate value theorem requires $f(a) < 0 < f(b)$ *strictly*, not weakly).
- Some are not: the classical Bolzano–Weierstrass theorem fails constructively without LPO (the limited principle of omniscience).

For ζ, the classical proofs of the functional equation and the prime number theorem use LEM in essential ways (compactness of contours, dichotomy in zero-detection). Whether constructive substitutes exist is an active open problem.

11.4 What the paper accomplishes

In summary:

- It identifies the comparison theorem as a precise statement, not a slogan.
- It provides the easy direction in two stages, with explicit stratification.
- It names the hard direction's missing components.
- It provides a concrete witness (functional equation) where the gap is visible.
- It frames Volume III's other results (OQ1, OQ2, OQ5, OQ6) as *partial answers to a programme* rather than as the programme itself, by making the comparison theorem the explicit goalpost against which they should eventually be measured.

This is the *honesty paper* of Volume III: without it, every HoTT-ζ result in the volume would carry the implicit assumption that HoTT-ζ is the same as classical ζ. With it, that assumption is named, scoped, and is itself the subject of Conjecture C.

12 Conclusion

We have addressed Open Question 4 of Volume III by formulating the foundational comparison theorem precisely, proving the forward direction in two stratified stages (Theorem A unconditionally for ζ-free formulas, Theorem B conditionally on (DC_ζ) for ζ-aware formulas), stating the reverse direction as Conjecture C with a full list of required infrastructure (I1)–(I5), and giving a concrete instance (Lemma D, the functional equation) where the forward direction succeeds and the reverse is open.

The paper does not establish the comparison theorem in full. It lays out exactly what would be required and what is currently missing. Volume III's remaining honesty is upheld: HoTT-ζ *may* be "really ζ," but until Conjecture C is proved, the assertion that it is is a research hypothesis, not a theorem.

Future work

1. Construct (I1) — a constructive Henkin model — for $\mathcal{L}_F[\zeta]$ with ζ_{HoTT} as forced interpretation.
2. Develop (I2) — a constructive completeness theorem for HoTT-internal first-order logic on Markov-stable fragments.
3. Use (I4) — completion of OQ2's analytic continuation library — to upgrade (DC_ζ) from conditional to unconditional across $\mathbb{C}_c \setminus \{1\}$.
4. Carry out (I5) — the LEM-detection analysis of the classical ζ proofs to identify reusable constructive substitutes.
5. Once (I1)–(I5) are in place, attempt Conjecture C's full proof.

Each of (1)–(5) is itself worth a paper.

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[1] *Note on dates.* Several entries below (LongVolII2026, LongOQ2, LoefflerStoll2025, *et al.*) refer to companion volumes of the present programme and to in-press / preprint material. Where a companion volume is in preparation, the citation should be read as "in preparation, YonedaAI Research Collective." The present preprint is part of Volume III of an ongoing multi-volume programme; cross-volume citations are stable identifiers within the programme rather than DOI-issued external publications.

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