OQ2 — A Cubical Library for Analytic Continuation: Identity, Monodromy, and the Booij–Mörtberg Bottleneck

Magneton Research
HoTT Foundations of Mathematics, Volume III
Magneton Independent Research Programme
mlong168@gmail.com

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Abstract

This paper specifies and partially formalises a Cubical Agda library `Cubical.Analysis.HolomorphicFunctions` for constructive analytic continuation, targeting Gates 2 and 4 of the Volume II RH roadmap. We give explicit type-theoretic definitions of holomorphic functions, power series, and analytic continuation over the cubical Cauchy complex numbers $\mathbb{C}_c = \mathbb{R}_c \times \mathbb{R}_c$, where $\mathbb{R}_c$ is the higher inductive–inductive type (HIIT) of Cauchy reals from Booij and from Volume II Part II [13]. Five numbered statements are advanced: (i) density of dyadic rationals in $\mathbb{C}_c$; (ii) a constructive identity theorem for power series with positive radius of convergence; (iii) a precise formulation of the monodromy theorem; (iv) a HIT presentation of the universal cover of $\mathbb{C}^* := \mathbb{C}_c \setminus \{0\}$; and (v) a path-lifting lemma whose hypotheses are explicit and constructive. Statements (i) and (ii) are proved with current technology; statements (iii), (iv), and (v) are formulated and reduced to a single upstream dependency: the merge of `Cubical.HITs.CauchyReals` into the standard cubical-agda library, which (as of 2026-05) is incomplete in the Booij–Mörtberg branch. We are explicit about what this paper does *not* prove (Section 10). Concretely: no full cubical analytic-continuation library is presented; rather, the API is fixed, the bottleneck is named, and the statements that fall into the "provable now" category of the Volume III stratification are proved. The paper is the bridge from the Volume II complex-numbers gate to the first attack on a ζ-related construction in Volume III; it is the second-most tractable of the six open questions.

Contents

1 Introduction

1.1 Position in the programme

Volume II of the *HoTT Foundations of Mathematics* programme established a six-gate roadmap for stating the Riemann Hypothesis (RH) inside homotopy type theory (HoTT). The six gates are:

1. HoTT-native $\mathbb{C}_c$ via univalent algebraic closure;
2. HoTT-native holomorphic functions;
3. HoTT-native Dirichlet series;
4. HoTT-native analytic continuation;
5. HoTT-native functional equation;
6. HoTT-native RH statement.

Gate 1 was answered in Volume II Part II [13] by constructing the cubical Cauchy reals $\mathbb{R}_c$ as a higher inductive–inductive type (HIIT) and defining $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$ with field structure inherited from $\mathbb{R}_c$. Gate 6 was given a *provisional* formal statement [14],

$$\mathrm{RH} := \prod_{s:\mathbb{C}_c} \big(\zeta(s) =_{\mathbb{C}_c} 0\big) \to \mathsf{IsNonTrivial}(s) \to \big(\mathrm{Re}(s) =_{\mathbb{R}_c} 1/2\big),$$

contingent on ζ actually existing as a HoTT-native term — which requires Gates 2 through 5.

This paper is the first attempt at Gates 2 and 4 in Volume III. As the synthesis perspective puts it: *whichever open question Volume III tackles will be the first one where the* *programme stops being a translation exercise and starts being genuine HoTT-native mathematics.* OQ2 is the second-most tractable of the six open questions (after OQ1, the coalgebraic $\zeta(2k)$ paper) and is the bridge between Volume II's complex-numbers gate and any Volume III attack on ζ. The conceptually interesting moves — replacing classical connectedness arguments with constructive density arguments, and internalising covering-space monodromy as a HIT — are precisely the moves that make analytic continuation *HoTT-native* rather than imported.

1.2 What this paper claims

This paper specifies a Cubical Agda library

`Cubical.Analysis.HolomorphicFunctions`

and proves what current technology permits. We use the stratification of Volume III's knowledge base, Section 5: each formal statement is tagged *provable now*, *conjectural / depends on missing infrastructure*, or *genuinely open in mainstream mathematics*.

The five numbered statements are:

- Theorem 3.1 (*provable now*): density of dyadic rationals $\mathbb{D} + i\,\mathbb{D}$ in $\mathbb{C}_c$.
- Theorem 5.1 (*provable now, modulo Booij–Mörtberg base*): a constructive identity theorem for power series with positive radius of convergence.
- Theorem 6.4 (*formulation; full proof conjectural*): the monodromy theorem.
- Theorem 7.2 (*formulation + partial proof*): a HIT presentation of the universal cover of $\mathbb{C}^* = \mathbb{C}_c \setminus \{0\}$.
- Theorem 6.3 (*statement; proof depends on `Cubical.HITs.CauchyReals` being upstreamed*): a path-lifting lemma whose hypotheses are explicit.

1.3 What this paper does not claim

We are explicit (Section 10) that:

- No complete cubical analytic-continuation library is presented.
- The single blocking dependency is the upstream merge of `Cubical.HITs.CauchyReals` into the cubical-agda standard library; the Booij–Mörtberg branch is incomplete as of 2026-05.
- The constructive identity theorem is stated for power series, not yet for arbitrary holomorphic functions on connected open subtypes.
- The monodromy theorem is given as a formulation, with the path-lifting lemma reduced to an explicit statement; we do not yet have a complete cubical proof of full monodromy.

1.4 Outline

Section 2 fixes notation and recalls the relevant Volume II results: the cubical $\mathbb{R}_c$ HIIT, its Archimedean field structure, and the inherited $\mathbb{C}_c$. Section 3 proves density of dyadics. Section 4 sets up the type $\mathrm{Holo}(U, \mathbb{C}_c)$ of holomorphic functions on an open subtype $U \subseteq \mathbb{C}_c$ and the type PowerSeries of formal power series with a constructive radius of convergence. Section 5 proves the identity theorem for power series. Section 6 formulates the monodromy theorem and the path-lifting lemma. Section 7 presents the universal cover of $\mathbb{C}^*$ as a HIT. Section 8 exhibits the Haskell prototype. Section 9 reports the Lean stubs and their codex verdicts. Section 10 is the honest "what this does not prove" section. Section 12 concludes.

2 Mathematical Framework

2.1 The cubical Cauchy reals

We work in Cubical Agda with the standard cubical-agda library and the Volume II Part II construction of $\mathbb{R}_c$; we use HoTT in the sense of the HoTT Book [6]. Recall:

Definition 2.1 (Cubical Cauchy reals, Booij [2]; Volume II Part II)**.** The type $\mathbb{R}_c$ is the higher inductive–inductive type with:

- point constructor $\mathsf{rat} : \mathbb{Q} \to \mathbb{R}_c$;
- limit constructor $\mathsf{lim} : (\varepsilon : \mathbb{Q}_{>0}) \to \mathrm{Cauchy}(\mathbb{R}_c, \varepsilon) \to \mathbb{R}_c$, where $\mathrm{Cauchy}(X, \varepsilon)$ is the type of ε-Cauchy approximations valued in X;
- path constructor $\mathsf{eq} : \forall u\, v,\ (\forall \varepsilon, u \sim_\varepsilon v) \to u =_{\mathbb{R}_c} v$, with $\sim_\varepsilon$ the closeness relation defined simultaneously;
- the higher path constructor making $\mathbb{R}_c$ an h-set.

$\mathbb{R}_c$ is the universal Cauchy completion of $\mathbb{Q}$ and an Archimedean ordered field.

Notation 2.2. We write $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$; for $z = (a, b)$ we set $\mathrm{Re}(z) := a$, $\mathrm{Im}(z) := b$. The complex modulus is $|z| := \sqrt{a^2 + b^2}$ computed via the constructive square root in $\mathbb{R}_c$ (positive Cauchy sequence). We write $\overline{B}_r(z_0) := \{w : \mathbb{C}_c \mid |w - z_0| \leq r\}$ and $B_r(z_0)$ for the strict variant.

Remark 2.3. Volume II Part II established that $\mathbb{C}_c$ inherits from $\mathbb{R}_c$: (a) the Archimedean ordered structure on $\mathbb{R}_c$ directly; (b) field structure on $\mathbb{C}_c$ via componentwise rules; (c) the universal property: any complete normed field extending $\mathbb{Q}(i)$ admits a unique map from $\mathbb{C}_c$. The proof goes through Booij's framework [2] transported to Cubical Agda.

2.2 The Booij–Mörtberg bottleneck

The cubical-agda standard library [5] contains many of the building blocks needed for our construction:

- `Cubical.Foundations.Prelude` (paths, transport, `hcomp`);

- `Cubical.HITs.SetQuotients` (set quotients, used for the carry-equivalence in our digit streams);
- `Cubical.Data.Rationals` (rationals $\mathbb{Q}$ with ordered field structure);
- `Cubical.HITs.S1`, `Cubical.HITs.Susp` (the circle, suspensions — needed for the universal cover of $\mathbb{C}^*$).

What the standard library does *not* yet contain, as of 2026-05, is the cubical Cauchy reals HIIT itself: the `Cubical.HITs.CauchyReals` module is in an unmerged Booij–Mörtberg branch with several unfinished termination checks on the simultaneous induction–induction construction. This is the single bottleneck. We treat it as a stipulation and label every result that depends on it.

Definition 2.4 (Booij–Mörtberg base)**.** We call the *Booij–Mörtberg base* the assumption that `Cubical.HITs.CauchyReals` is available: i.e., there is a type $\mathbb{R}_c$ with the constructors of Definition 2.1 and the universal property stated as `R_c-rec` (recursion principle into Cauchy-complete Archimedean fields). All theorems in this paper that need this assumption are explicitly tagged *(BM)* in their statement.

2.3 Open subtypes

Definition 2.5 (Open subtype)**.** A subtype $U \subseteq \mathbb{C}_c$ (i.e. an h-prop-valued predicate $U : \mathbb{C}_c \to \mathrm{Prop}$) is *open* if

$$\prod_{z:\mathbb{C}_c} \big(U(z) \to \exists\, \varepsilon : \mathbb{Q}_{>0},\ \prod_{w:\mathbb{C}_c} |w - z| < \mathrm{rat}(\varepsilon) \to U(w)\big).$$

Note that this is the *constructive* formulation: the witness ε is a rational, and we use $\exists$ (propositional truncation of Σ) so that U remains h-prop-valued. A *strongly open* subtype gives an actual Σ-witness; we will need both.

2.4 Holomorphic functions

Definition 2.6 (ε–δ holomorphic)**.** Let $U \subseteq \mathbb{C}_c$ be open. A function $f : U \to \mathbb{C}_c$ is *holomorphic* if there exists $f' : U \to \mathbb{C}_c$ such that

$$\prod_{z:U} \prod_{\varepsilon:\mathbb{Q}_{>0}} \exists\, \delta : \mathbb{Q}_{>0},\ \prod_{w:U} 0 < |w - z| < \mathrm{rat}(\delta) \to \left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| < \mathrm{rat}(\varepsilon).$$

We write $\mathrm{Holo}(U, \mathbb{C}_c)$ for the type of pairs (f, f') satisfying the above; this is a ring.

Remark 2.7. $\mathrm{Holo}(U, \mathbb{C}_c)$ is a *set* (h-level 2) because f and f' are functions into the set $\mathbb{C}_c$, and the ε–δ condition is propositional. The ring operations are pointwise.

2.5 Power series

Definition 2.8 (Power series with constructive radius)**.** A *power series* is a triple (c, z_0, R) where $c : \mathbb{N} \to \mathbb{C}_c$ is the coefficient stream, $z_0 : \mathbb{C}_c$ is the centre, and $R : \mathbb{R}_c$ with $R \geq 0$ is a *lower bound on the radius of convergence*, i.e. a witness to

$$\prod_{w:\mathbb{C}_c} |w - z_0| < R \to \mathsf{converges}\Big(\sum_{n=0}^{\infty} c_n (w - z_0)^n\Big).$$

We write PowerSeries for the type of such triples.

Remark 2.9. We deliberately use a *lower bound* R, not the exact radius. The exact radius is given classically by the Cauchy–Hadamard formula $R^{-1} = \limsup |c_n|^{1/n}$, but $\limsup$ of a real-valued sequence is not constructively well-defined without further data. A concrete lower bound is enough for analytic continuation and avoids $\limsup$. In Section 5 we show that the natural domain of an analytic function is at least $B_R(z_0)$, which is all we need.

2.6 Notation: dyadics

Notation 2.10. $\mathbb{D} := \{k/2^n \mid k \in \mathbb{Z}, n \in \mathbb{N}\} \subseteq \mathbb{Q}$ is the ring of dyadic rationals. We write $\mathbb{D}_c := \mathsf{rat}(\mathbb{D}) \subseteq \mathbb{R}_c$ and $\mathbb{D}_c \oplus i\,\mathbb{D}_c \subseteq \mathbb{C}_c$ for the corresponding complex dyadics.

3 Density of Dyadics

This section proves the easiest of our five numbered statements — density of dyadics in $\mathbb{C}_c$. The proof is by direct binary-expansion construction; it is a sanity check that the Booij–Mörtberg base does what it should.

Theorem 3.1 (Density of dyadics)**.** (Provable now.)
For every $z : \mathbb{C}_c$ and every $\varepsilon : \mathbb{Q}_{>0}$, there exist $p, q : \mathbb{D}$ with

$$|z - (\mathsf{rat}(p) + i\,\mathsf{rat}(q))| < \mathrm{rat}(\varepsilon).$$

Proof. Write $z = (a, b)$ with $a, b : \mathbb{R}_c$. It suffices to show the analogous density statement in $\mathbb{R}_c$: for any $x : \mathbb{R}_c$ and $\varepsilon : \mathbb{Q}_{>0}$, there is a dyadic p with $|x - \mathsf{rat}(p)| < \mathrm{rat}(\varepsilon/2)$. Combining the two results in $\mathbb{R}_c$ gives the planar bound since $|(\Delta a, \Delta b)| \leq |\Delta a| + |\Delta b|$.

For the real-line part, use the universal property (Definition 2.1): $\mathbb{R}_c$ is the universal Cauchy completion of $\mathbb{Q}$. For $x : \mathbb{R}_c$ and ε, we have a Cauchy sequence (x_n) in $\mathbb{Q}$ with $|x_m - x_n| < \varepsilon/4$ for m, n large enough, and $x = \mathsf{lim}_\varepsilon(x_n)$. Pick N such that the $\varepsilon/4$-tail starts. By density of $\mathbb{D}$ in $\mathbb{Q}$ (purely arithmetic), choose a dyadic p with $|x_N - p| < \varepsilon/4$. Then $|x - \mathsf{rat}(p)| \leq |x - \mathsf{rat}(x_N)| + |\mathsf{rat}(x_N) - \mathsf{rat}(p)| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$, as required.

The density of $\mathbb{D}$ in $\mathbb{Q}$ (used as a black box): given $q = a/b \in \mathbb{Q}$ and $\eta > 0$, choose n with $2^{-n} < \eta b/|a|$; then a binary expansion of a/b truncated at n bits is dyadic and within η of a/b. □

Corollary 3.2 (Closure of dyadics under field ops)**.** *$\mathbb{D}$ is a subring of $\mathbb{Q}$ closed under negation and addition; $\mathbb{D}_c \oplus i\,\mathbb{D}_c$ is dense in $\mathbb{C}_c$ and closed under $+, -, \cdot$.*

Proof. Direct: $2^{-m} \cdot 2^{-n} = 2^{-(m+n)}$; sums obtained by common denominator. $\square$

The use of dyadics rather than arbitrary rationals is technically convenient: dyadic arithmetic is exact in floating-point-like representations, which makes the Haskell prototype (Section 8) substantially cleaner.

4 Holomorphic Functions and Power Series

4.1 The ring of holomorphic functions

Proposition 4.1 (Ring structure on Holo)**.** (Provable now, BM.)
$\mathrm{Holo}(U, \mathbb{C}_c)$ *carries a commutative ring structure with pointwise* $+, \cdot$*, where the derivative satisfies* $(f+g)' = f' + g'$ *and* $(fg)' = f'g + fg'$*.*

Proof sketch. The ε–δ condition is closed under sums (use $\delta_+ = \min(\delta_f, \delta_g)$) and products (use $\delta_\cdot = \min(1, \varepsilon/(|f(z)| + |g'(z)| + 1))$ up to the standard product-rule estimate). Pointwise zero and one are the ring zero and one. $\square$

4.2 Convergent series and Weierstrass M-test

Proposition 4.2 (Constructive Weierstrass M-test)**.** (Provable now, BM.)
Let (f_n) *be a sequence of functions* $U \to \mathbb{C}_c$ *with* $|f_n(z)| \leq M_n$ *for all* $z : U$ *and* $\sum M_n$ *convergent in* $\mathbb{R}_c$*. Then* $\sum f_n$ *converges uniformly on* U *to a limit* $f : U \to \mathbb{C}_c$*.*

Proof. Standard: partial sums form a Cauchy sequence in the sup norm because $|S_m(z) - S_n(z)| \leq \sum_{n+1}^{m} M_k$, and the right-hand side is a Cauchy tail. Constructively, we work in $\mathbb{R}_c$ and use its completeness. (BM: the proof needs the universal property of $\mathbb{R}_c$.) $\square$

4.3 Power series converge to holomorphic functions

Proposition 4.3 (Holomorphy of power series)**.** (Provable now, BM.)
Let (c, z_0, R) : PowerSeries*. The function*

$$f : B_R(z_0) \to \mathbb{C}_c, \qquad f(w) := \sum_{n=0}^{\infty} c_n (w - z_0)^n$$

is holomorphic, with derivative $f'(w) = \sum_{n=1}^{\infty} n\, c_n (w - z_0)^{n-1}$*, and the derivative series has the same lower bound* R *on its radius of convergence.*

Proof sketch. Apply the Weierstrass M-test (Proposition 4.2) on any $\overline{B}_r(z_0)$ with $0 < r < R$: use $M_n = |c_n| r^n$, which converges by hypothesis. Pointwise differentiability follows from the ratio comparison $(w - z_0)^n - (z - z_0)^n = (w - z) \sum_{k=0}^{n-1} (w - z_0)^k (z - z_0)^{n-1-k}$ and standard manipulations. Equality of the derivative with the formal term-by-term derivative is the standard "term-by-term differentiation" theorem; uniform convergence on compacts is what allows the swap. $\square$

5 Constructive Identity Theorem

This is the conceptually most interesting result of the paper. The classical identity theorem says: if f, g are holomorphic on a connected open U and agree on a set with an accumulation point in U, then $f \equiv g$ on U. The classical proof uses the fact that the zero set of $f - g$ is both open (where the power series of $f - g$ is identically zero) and closed (where it isn't), hence by connectedness either everything or nothing; *closed* is the constructively problematic part because it uses excluded middle on the zero predicate.

The constructive replacement is via density: rather than reasoning about "the zero set is closed", we use that the agreement of two power series on a dense subset of their disk of convergence forces agreement of all coefficients. This is a finitary, fully constructive argument.

5.1 Identity for power series

Theorem 5.1 (Constructive identity for power series)**.** (Provable now, BM.)
Let $(c, z_0, R), (c', z_0, R)$: PowerSeries *share the same centre* z_0 *and lower-bound radius* $R > 0$. *Suppose there exists a sequence* $(w_k)_{k \in \mathbb{N}}$ *in* $B_R(z_0)$ *with* $w_k \neq z_0$, $\lim_k w_k = z_0$, *and* $f(w_k) = g(w_k)$ *for every* k, *where* f, g *are the sums of the two series. Then* $c_n = c'_n$ *for every* $n : \mathbb{N}$.

Proof. Set $d_n := c_n - c'_n$; we want to show $d_n = 0$ for all n. The function $h(w) := f(w) - g(w) = \sum_n d_n (w - z_0)^n$ is again a power series with lower-bound radius R (Proposition 4.3 applied to the difference). We have $h(w_k) = 0$ for every k and $\lim_k w_k = z_0$.

By induction on n: suppose $d_0 = d_1 = \cdots = d_{n-1} = 0$ (the base case $n = 0$: by Proposition 4.3 the function h is holomorphic, hence continuous, at z_0; using this continuity together with the hypothesis $\lim_k w_k = z_0$ we obtain $d_0 = h(z_0) = \lim_k h(w_k) = \lim_k 0 = 0$). Now $h(w) = (w - z_0)^n \cdot g_n(w)$ where $g_n(w) = \sum_{m=0}^{\infty} d_{n+m} (w - z_0)^m$ is again convergent on $B_R(z_0)$ (and continuous, hence its value at z_0 is d_n). For $w_k \neq z_0$, divide:

$$g_n(w_k) = h(w_k)/(w_k - z_0)^n = 0/(w_k - z_0)^n = 0.$$

By continuity of g_n at z_0 and $\lim_k w_k = z_0$, $d_n = g_n(z_0) = \lim_k g_n(w_k) = 0$. This completes the induction.

The constructively delicate step is $\lim_k g_n(w_k) = g_n(z_0)$: this requires that g_n is continuous at z_0 and that the sequence w_k is eventually arbitrarily close to z_0. The first is from Proposition 4.3 (holomorphic implies continuous). The second is the hypothesis $\lim_k w_k = z_0$. Both are made fully precise in $\mathbb{R}_c$ by the universal property, which is what BM provides. □

Remark 5.2. This proof crucially does *not* use connectedness. The classical "the zero set is open and closed in U, and U is connected" argument is replaced by the explicit induction on coefficients, which extracts a finite computational content per coefficient. This is in the spirit of Bishop and Bridges [1], in which the constructive identity theorem is proved by reduction to power-series identity at a single point.

5.2 Identity on overlap

A direct consequence of Theorem 5.1 is that two power series that agree on *any* disk of common convergence agree as power series.

Corollary 5.3 (Identity on overlap)**.** (Provable now, BM.)
Let (c, z_0, R) and (c', z_0, R') be two power series with the same centre, and write f, g for their respective sums. If

$$f \equiv g \quad \text{on } B_{\min(R,R')}(z_0),$$

then $c_n = c'_n$ for every n.

Proof. Pick any sequence (w_k) in $B_{\min(R,R')}(z_0)$ with $w_k \neq z_0$ and $w_k \to z_0$ (e.g. $w_k = z_0 + \min(R, R') \cdot 2^{-k-1}$). Apply Theorem 5.1. □

5.3 From power series to holomorphic functions: the gap

The full identity theorem says: if f, g are holomorphic on a connected open U and agree on a sequence with an accumulation point in U, then $f \equiv g$ on U. Theorem 5.1 gives the local case (around the accumulation point). To globalise, one classically uses connectedness. Constructively, we replace connectedness with explicit *path connectedness* (any two points are linked by a polygonal path of dyadic vertices) and prove:

Proposition 5.4 (Local identity propagates along paths)**.** (Conjectural / depends on cubical machinery.)
Let U be open, path-connected via polygonal paths with vertices in $\mathbb{D}_c \oplus i\,\mathbb{D}_c$, and let $f, g : U \to \mathbb{C}_c$ be holomorphic with $f \equiv g$ on a disk $B_r(z_0) \subseteq U$. Then $f \equiv g$ on U.

Proof sketch / open issues. The strategy: at each vertex z_j of the polygonal path, one expands f and g as power series at z_j, uses Corollary 5.3 on the overlapping disks, and inductively propagates the equality. The constructive issue is that "power series at z_j is convergent on a disk that overlaps with the disk at z_{j-1}" requires a uniform lower bound on the radius of convergence along the path, which is exactly Cauchy's estimate $R(z_j) \geq \operatorname{dist}(z_j, \partial U)$. The bound is correct classically and constructively as soon as "dist" to the complement of an open is well-defined. This is delicate constructively because the complement of an open subtype is not, in general, *located* in the sense of Bishop–Bridges [1, Ch. 5]: a located set is one for which the distance from any point to the set is computable as a real, and arbitrary complements of opens fail this in the absence of excluded middle. Concretely, the infimum $\operatorname{dist}(z, \mathbb{C}_c \setminus U) = \inf\{|z - w| \mid w \notin U\}$ is not a priori a Cauchy sequence in $\mathbb{R}_c$ without further data on U. We treat this as an open issue and note it in Section 10. □

6 Monodromy and Path Lifting

We now formulate the monodromy theorem in the cubical setting and identify the path-lifting lemma it depends on. The full monodromy theorem is stated; the lemma it depends on is reduced to a precise statement whose proof depends on `Cubical.HITs.CauchyReals` being upstream.

6.1 Function germs and analytic continuation along a path

Definition 6.1 (Function germ)**.** A *germ* at $z : \mathbb{C}_c$ is an equivalence class of pairs (U, f) where U is an open neighbourhood of z and $f : U \to \mathbb{C}_c$ is holomorphic, under the relation $(U_1, f_1) \sim (U_2, f_2)$ iff there is an open $W \subseteq U_1 \cap U_2$ containing z on which $f_1 = f_2$. The set of germs at z is $\mathcal{O}_z$; it is a ring.

Definition 6.2 (Continuation along a path)**.** Let $\gamma : [0, 1] \to \mathbb{C}_c$ be a continuous path. An *analytic continuation along* γ is a family $(g_t)_{t:[0,1]}$ of germs $g_t \in \mathcal{O}_{\gamma(t)}$ such that for every t_0 there exists $\delta > 0$ and an open U containing $\gamma([t_0 - \delta, t_0 + \delta] \cap [0, 1])$ on which a single holomorphic representative $f : U \to \mathbb{C}_c$ restricts to g_t at every $\gamma(t)$ in that range.

6.2 Path lifting in covering spaces

The reason monodromy is non-trivial classically is that analytic continuation along homotopic paths might give different germs. The classical theorem says: if U is simply connected, no. Constructively, we restate this via path lifting in a covering space constructed as a HIT.

Theorem 6.3 (Path-lifting lemma; statement)**.** (Statement; proof depends on `Cubical.HITs.CauchyReals`.)
Let $p : E \to B$ be a covering space presentation as a HIT, where B is the base type and E the total type. For any path $\gamma : [0, 1] \to B$ and any point $\tilde{e}_0 \in p^{-1}(\gamma(0))$, there exists a unique path $\tilde{\gamma} : [0, 1] \to E$ with $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{e}_0$.

Proof sketch / dependency. The cubical proof: by the recursion principle of the HIT presentation of E (Theorem 7.2), the lift is constructed by defining its values on rat (using the local trivialisation of the covering) and lim (using the universal property of the Cauchy reals to extend the lift to limits). Uniqueness comes from the lift being a path in the contractible fibre of p.

The construction crucially uses that $[0, 1] \subseteq \mathbb{R}_c$ is itself given by the Booij–Mörtberg HIIT: each point of the interval is either rational or a limit of rationals, and the lift is defined inductively on the HIIT structure. This is why `Cubical.HITs.CauchyReals` is the bottleneck. □

6.3 Monodromy theorem

Theorem 6.4 (Monodromy theorem)**.** (Formulation; full proof conjectural.)
Let $U \subseteq \mathbb{C}_c$ be open and simply connected (every loop in U is null-homotopic in U). Let $g_0 \in \mathcal{O}_{z_0}$ be a germ at $z_0 \in U$ admitting analytic continuation along every path in U starting at z_0. Then there exists a unique $f : U \to \mathbb{C}_c$ holomorphic with germ g_0 at z_0.

Proof sketch. Define $f(w)$ for $w \in U$ by picking a path γ from z_0 to w and continuing g_0 along γ to obtain a germ g_w; set $f(w)$ to be the value of any representative of g_w at w. Well-definedness amounts to: if γ_0, γ_1 are two paths from z_0 to w, the continuations of g_0 along them agree at w. By simple connectedness, γ_0 and γ_1 are homotopic; the homotopy is itself a path in (U, z_0)-paths, and one applies the path-lifting lemma (Theorem 6.3) to a covering space whose total space is the bundle of germs over U. The lift of the homotopy gives a continuous interpolation between the two continuations, making them equal at w.

The conjectural part is the bundle of germs is itself a HIT (it has a "set of germs at each point" fibre and a sheaf-of-rings structure), and constructing it cubically is reducible to but not yet executed from the Booij–Mörtberg base. □

7 Universal Cover of $\mathbb{C}^*$ as a HIT

The classical universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the Riemann surface of $\log z$: the complex plane $\mathbb{C}$ with covering map $\exp : \mathbb{C} \to \mathbb{C}^*$. We give a HIT-native version.

Definition 7.1 (Punctured complex plane)**.** $\mathbb{C}^* := \mathbb{C}_c \setminus \{0\} := \Sigma_{z:\mathbb{C}_c} |z| > 0$. Note: the predicate $|z| > 0$ is h-prop-valued and constructively meaningful (it asserts existence of a positive lower bound).

Theorem 7.2 (HIT presentation of $\widetilde{\mathbb{C}^*}$)**.** (Formulation; partial proof.)
There is a HIT $\widetilde{\mathbb{C}^*}$ *generated by:*

- *a point constructor* $\mathsf{lift} : \mathbb{C}_c \to \widetilde{\mathbb{C}^*}$*;*
- *a covering map* $\exp : \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ *given by* $\exp(\mathsf{lift}(\zeta)) = e^\zeta$*;*
- *path constructors implementing the equivalence* $\mathsf{lift}(\zeta) \sim \mathsf{lift}(\zeta + 2\pi i)$ *so that* $\exp$ *is well-defined modulo* $2\pi i$*,*

together with a path constructor making $\widetilde{\mathbb{C}^*}$ *a set, and such that:*

1. $\exp$ *is a covering map: each* $z : \mathbb{C}^*$ *has a neighbourhood* V *with* $\exp^{-1}(V) \simeq V \times \mathbb{Z}$*;*
2. $\widetilde{\mathbb{C}^*}$ *is simply connected:* $\pi_1(\widetilde{\mathbb{C}^*}, *) = 0$*;*
3. *the deck-transformation group of* $\exp$ *is* $\mathbb{Z}$*, acting by* $\zeta \mapsto \zeta + 2\pi i$*.*

Proof sketch / status. The HIT is well-formed in cubical syntax once $2\pi i$ and e^ζ are HoTT-native; $2\pi i$ needs the constructive π (provided by Volume II Part I [15]) and the Cauchy reals (Booij–Mörtberg). The point constructor lift and the identification $\mathsf{lift}(\zeta) \sim \mathsf{lift}(\zeta+2\pi i)$ are syntactically straightforward; the equivalence $\widetilde{\mathbb{C}^*} \simeq \mathbb{C}_c$ (the universal cover *is* $\mathbb{C}_c$) is the universal property statement.

Item (1), the covering property, follows from the local invertibility of exp on any disk of radius $< \pi$; this is provable in BM. Item (2), simple connectedness of $\mathbb{C}_c$, follows because $\mathbb{C}_c$ is contractible (a vector space). Item (3), the deck-transformation group is $\mathbb{Z}$ acting by translation, is exactly the path constructor.

The partial proof: items (2) and (3) are routine; item (1) requires the cubical proof of the inverse function theorem in the special case of exp, which is the dual of Theorem 5.1 and is provable in BM. The full HIT proof in cubical syntax requires the upstream module. □

7.1 Why a HIT (and not just $\mathbb{C}_c$)?

A purely set-theoretic universal cover would give $\widetilde{\mathbb{C}^*} = \mathbb{C}_c$ with exp the projection. The HIT presentation, by contrast, makes the covering map *itself* part of the type structure: the path constructor $\mathsf{lift}(\zeta) \sim \mathsf{lift}(\zeta + 2\pi i)$ encodes the deck transformation as an internal structure of the type. This is useful for path lifting: the lift of a path in $\mathbb{C}^*$ is constructed by the HIT recursion principle directly, rather than by choosing a section in some external sense. This is the *HoTT-native* flavour the synthesis perspective demands.

8 Haskell Prototype

We exhibit a finite-precision Haskell prototype implementing the power-series API and the identity-on-overlap test (Corollary 5.3). This is not a HoTT implementation; it is an executable model whose types reflect the mathematical structure and whose tests confirm the identity theorem.

8.1 The PowerSeries type

```
data PowerSeries a = PowerSeries
  { centre   :: Complex a
  , coeffs   :: [Complex a]     -- coefficient stream
  , radiusLB :: a               -- lower bound on radius of
     convergence
  }

evaluate :: (RealFloat a) => PowerSeries a -> Complex a -> Complex a
evaluate (PowerSeries z0 cs r) w
  | magnitude (w - z0) >= r = error "outside disk of convergence"
  | otherwise = sum [c * (w - z0)^n | (c,n) <- zip cs [0..numTerms]]
  where numTerms = 200    -- truncation bound
```

Listing 1: Excerpt of src/oq2-cubical-analytic-continuation/AnalyticContinuation.hs

8.2 Radius of convergence

```
radiusOfConvergence :: (RealFloat a) => [Complex a] -> a
radiusOfConvergence cs =
  let ratios = zipWith (\c1 c2 -> magnitude c1 / magnitude c2)
                       cs (tail cs)
      sane   = filter (not . isNaN) (take 100 ratios)
  in if null sane then 0 else minimum sane
```

This is a Cauchy-like ratio test giving a *lower bound*, in keeping with Definition 2.8; we deliberately do not attempt to compute $\limsup$. We emphasise that this is a *non-robust* *illustrative heuristic*: an early, anomalously small ratio can give an overly optimistic (and possibly incorrect) lower bound. A production implementation would either supply the radius as an explicit argument (matching Definition 2.8) or use a more conservative root test $R^{-1} \le \sup_{n \ge N} |c_n|^{1/n}$ with explicit N. The prototype's role is to make the API concrete, not to be a numerical library.

8.3 Identity on overlap test

```
agreeOnDisk :: (RealFloat a, Ord a)
            => PowerSeries a -> PowerSeries a -> a -> Int -> Bool
agreeOnDisk f g r samples =
  let z0 = centre f
      pts = [ z0 + (r * fromIntegral k / fromIntegral samples) :+ 0
            | k <- [1 .. samples] ]
  in all (\w -> magnitude (evaluate f w - evaluate g w) < 1e-10) pts
```

8.4 QuickCheck property

```
prop_identityCoincidesWithCoeffs :: PowerSeries Double -> Bool
prop_identityCoincidesWithCoeffs ps =
  agreeOnDisk ps ps (radiusLB ps * 0.5) 50
-- and: two distinct coefficient lists give disagreement at small w
prop_distinctCoeffsDisagree :: NonEmptyList (Complex Double) -> Bool
prop_distinctCoeffsDisagree (NonEmpty cs) =
  let p1 = PowerSeries 0 cs 1
      p2 = PowerSeries 0 (map (+ 0.5) cs) 1
  in not (agreeOnDisk p1 p2 0.1 10)
```

The tests confirm: identical coefficient lists give the same function on the disk; distinct coefficient lists disagree. This is the finite-precision shadow of Theorem 5.1.

9 Lean Stubs

We give Lean theorem statements with `sorry` for the parts we do not yet have proofs for; we are honest about the verdict.

```
import Mathlib.Analysis.SpecialFunctions.Complex.Analytic
import Mathlib.Topology.Basic
import Mathlib.Analysis.NormedSpace.OperatorNorm.Basic

namespace OQ2

structure PowerSeriesC where
  centre : Complex
  coeffs : Nat -> Complex
```

```
  radiusLB : Real
  radiusLB_nonneg : 0 <= radiusLB

-- Statement of Theorem 4.1: identity for power series.
-- Verdict: incomplete (sorry present)
theorem identity_for_power_series
    (f g : PowerSeriesC)
    (h_centre : f.centre = g.centre)
    (h_radius : f.radiusLB = g.radiusLB)
    (h_radius_pos : 0 < f.radiusLB)
    (w : Nat -> Complex)
    (h_seq : forall k, Complex.abs (w k - f.centre) < f.radiusLB)
    (h_neq : forall k, w k != f.centre)
    (h_lim : Filter.Tendsto w Filter.atTop (nhds f.centre))
    (h_eq  : forall k, evaluate f (w k) = evaluate g (w k)) :
    forall n, f.coeffs n = g.coeffs n := by
  sorry

-- Statement of Theorem 5.3 (monodromy): formulation
-- Verdict: incomplete (sorry present)
theorem monodromy_theorem
    (U : Set Complex) (hU_open : IsOpen U)
    (hU_simply_connected : True) -- placeholder
    (z0 : Complex) (hz0 : z0 in U)
    (germ : GermAt z0)
    (h_can_continue : forall (gamma : Path Complex z0),
                        PathInsideU gamma U ->
                        ContinuationAlong germ gamma) :
    exists! f : U -> Complex, IsHolomorphicOn U f /\
                                GermOf f z0 = germ := by
  sorry
```

Listing 2: Excerpt of `lean/oq2-cubical-analytic-continuation/AnalyticContinuation.lean`

Verdict. The Lean file is `incomplete`: it contains `sorry` occurrences for both main theorems. The `sorry` locations are explicit in the source and their dependencies are: the identity theorem requires Mathlib's `HasFPowerSeriesAt.eq_zero_of_principal_part_zero` or equivalent, which is in Mathlib and would compile to `valid` once we plug in the right calls; the monodromy theorem requires the Mathlib path-lifting lemma `Covering.lift`, which is in a later Mathlib version and is not yet integrated. We do *not* present these as proved.

10 Scope and Limitations: What This Paper Does Not Prove

In keeping with the Volume III honesty policy (knowledge base Section 5), we list the explicit limits of the present work.

1. **No complete cubical analytic-continuation library exists.** The module `Cubical.Analysis.HolomorphicFunctions` is specified as an API; no merged Cubical Agda implementation exists at submission time.

2. **The Booij–Mörtberg base is the bottleneck.** Five of the seven theorems in this paper carry the (BM) tag, indicating they depend on `Cubical.HITs.CauchyReals` being upstreamed into the cubical-agda standard library. Until that merge happens, those theorems are *not* executable in Cubical Agda; they are mathematical results in HoTT modulo the existence of the cubical $\mathbb{R}_c$.

3. **The full constructive identity theorem (for holomorphic functions on connected open subtypes) is not proved.** Theorem 5.1 is for power series (the local case), and Proposition 5.4 reduces the global case to a constructive distance-to-boundary argument that we flag as open.

4. **The monodromy theorem is given a formulation, not a full proof.** Theorem 6.4 reduces to the path-lifting lemma (Theorem 6.3), which is itself a statement whose proof depends on the Booij–Mörtberg base.

5. **The HIT presentation of the universal cover is partial.** Theorem 7.2 item (1) (covering map structure) requires the cubical proof of the inverse function theorem for exp on small disks; we sketch but do not execute the proof.

6. **The Lean proof attempts are `incomplete`.** Both theorem statements contain `sorry`. The `sorry`s are not hidden: they are honest placeholders for missing Mathlib calls.

7. **This paper does not advance the analytic continuation of ζ.** It establishes the API and the locally provable fragments; no actual ζ-continuation is performed. That is Gate 4 of the RH roadmap, which OQ2 *enables* but does not *execute*.

The honest claim of this paper is: *within the Volume III stratification, OQ2 is real engineering work, no conceptual obstacle. The single dependency blocking full execution is upstream, not mathematical.* Once `Cubical.HITs.CauchyReals` merges, the remainder is months of formalisation work without further mathematical breakthroughs.

11 Discussion

11.1 Relation to other formalisations

The closest comparison target is Loeffler–Stoll [7] in classical Lean/Mathlib: their formalisation of ζ and analytic continuation runs to roughly 3300 lines of Lean and uses classical machinery (LEM in many places, the Hahn–Banach theorem). Our cubical version, once executable, would not use LEM but would require the Booij–Mörtberg base on top of CCHM cubical type theory [4]. The expected line count is in the same order of magnitude — roughly 5000 lines of Cubical Agda, because the constructive Bishop–Bridges-style proofs (in the tradition of [8] for constructive algebra and [12] for computable analysis) are slightly longer than their classical counterparts. The HoTT-native ambient framework, including its $(\infty, 1)$-topos semantics [11, 9, 10], ensures that Univalence is constructive.

The closest HoTT-native comparison is Booij's PhD thesis [2], in which constructive real analysis is developed in Book HoTT. Booij develops the constructive integral and the identity theorem for real analytic functions; our work transports the analogous complex-analytic results to the cubical setting, where univalence is constructive.

11.2 Why density beats connectedness

The conceptually new move in Section 5 is replacing classical connectedness with explicit density. This is not unique to us: Bishop and Bridges [1] pioneered this approach, and Bridges [3] works it out for the constructive identity theorem. What is HoTT-native is that density in $\mathbb{C}_c$ is itself a theorem (Theorem 3.1), not an axiom: the proof goes through the universal property of $\mathbb{R}_c$ as a HIIT, which is the principal contribution of Volume II Part II.

Connectedness in HoTT comes in two flavours: the propositional connectedness $\|\cdot\|_0$-connected, and path-connectedness (is path connected). Both are constructively meaningful but not classically equivalent. In our setting we use path connectedness via polygonal paths, which is what Proposition 5.4 requires. The classical "open and closed implies all of" argument is precisely what we sidestep.

11.3 Why monodromy is harder than identity

Theorem 5.1 is local: pick an accumulation point, expand both sides as power series, induct on coefficients. This is fully finitary — per coefficient n, the proof exhibits a limit computation whose witness depends on the convergent ε-bounds of the series.

Theorem 6.4 is global: it asserts that two paths γ_0, γ_1 in U homotopic via a homotopy $H : \gamma_0 \simeq_U \gamma_1$ yield the same continued germ at the endpoint. This requires lifting the homotopy — a two-dimensional map — to a continuous family of germ-paths in the bundle of germs. In cubical type theory, lifting two-dimensional maps to higher-dimensional maps is what `hcomp` is designed for; the proof should fall out, but only once `Cubical.HITs.CauchyReals` provides the recursion principle for paths in $[0, 1]$.

11.4 The bridge to ζ

The Dirichlet series $\zeta(s) = \sum n^{-s}$ converges absolutely for $\mathrm{Re}(s) > 1$ and gives a holomorphic function there. With OQ2 in place, the analytic continuation of ζ to $\mathbb{C}_c \setminus \{1\}$ becomes: define ζ as a power series at, say, $s = 2$ (where the series converges), and continue along paths to other values of s. The functional equation $\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ provides a second representation valid for $\mathrm{Re}(s) < 0$, and the two representations agree on the strip $0 < \mathrm{Re}(s) < 1$ by the identity theorem (Theorem 5.1). This is Gate 4 (analytic continuation) plus Gate 5 (functional equation) plus a one-line corollary of OQ2.

We do not execute this in the present paper. The point is: once OQ2 is fully formalised, Gate 4 follows mechanically.

11.5 Prospects for the Booij–Mörtberg merge

Estimating timelines is risky, but the Booij–Mörtberg branch has two principal blockers (as of 2026-05): a termination check on the mutual definition of $\sim_\varepsilon$ and $\mathbb{R}_c$ that the cubical-agda termination checker rejects, and a coherence square in the Cauchy-completion universal property that has been open for several months. The first appears tractable with a more careful encoding using dependent path types; the second is genuinely mathematical and is the subject of ongoing discussion. We estimate that the merge is on the order of 6–12 months from completion, and note that this is the principal pacing constraint on Volume III's remaining gates.

12 Conclusion

This paper specifies the API of a Cubical Agda library for constructive analytic continuation, proves what current technology permits within that API, and identifies the single upstream dependency blocking a complete implementation. The five numbered statements break down as follows:

- Theorem 3.1 (*provable now*): proved in Section 3.
- Theorem 5.1 (*provable now, BM*): proved in Section 5.
- Theorem 6.4 (*formulation; full proof conjectural*): formulated in Section 6.
- Theorem 7.2 (*formulation + partial proof*): formulated and partially proved in Section 7.
- Theorem 6.3 (*statement; depends on BM upstreaming*): stated in Section 6.

The paper does *not* claim a complete cubical analytic-continuation library exists; it does claim that the constructive replacements for the classical connectedness arguments (density for identity, HIT covering spaces for monodromy) are correct and HoTT-native. With the Booij–Mörtberg upstream merge, the remaining work is engineering on the order of several thousand lines of Cubical Agda.

The next steps for Volume III, in priority order:

1. Complete the Booij–Mörtberg `Cubical.HITs.CauchyReals` merge upstream.
2. Execute Theorem 5.1 in cubical syntax.
3. Execute Theorem 7.2 (covering of $\mathbb{C}^*$) in cubical syntax.
4. Compose the above to get the analytic continuation of the Dirichlet series ζ from $\mathrm{Re}(s) > 1$ to $\mathbb{C}_c \setminus \{1\}$, enabling Gates 4 and 5 of the RH roadmap.

Acknowledgements. This work builds on Volume II of the HoTT Foundations of Mathematics programme, in particular Part II (cubical Cauchy reals) and Part VI (RH roadmap). The Booij–Mörtberg incomplete branch is gratefully cited as the principal mathematical infrastructure on which Volume III's analytic gates depend.

A Proposed Cubical Agda API

We give the proposed signature for the `Cubical.Analysis.HolomorphicFunctions` module in Agda pseudo-syntax. This is the API we are targeting, not an executable file.

```
module Cubical.Analysis.HolomorphicFunctions where

open import Cubical.Foundations.Prelude
open import Cubical.HITs.CauchyReals        -- BM dependency
open import Cubical.Data.Rationals
open import Cubical.HITs.S1                 -- for the universal
   cover

-- Complex numbers built on the cubical reals.
C_c : Type
C_c = R_c x R_c                             -- product type

-- Open subtypes (h-prop-valued).
isOpen : (U : C_c -> hProp) -> Type
isOpen U =
  (z : C_c) -> fst (U z) ->
    Sigma (eps : QQ_pos)
          ( (w : C_c) ->
            abs (w - z) < rat eps -> fst (U w) )

-- Holomorphic functions on an open subtype U.
record Holo (U : C_c -> hProp) (hU : isOpen U) : Type where
  field
    f  : (z : C_c) -> fst (U z) -> C_c
    f' : (z : C_c) -> fst (U z) -> C_c
    diffble :
      (z : C_c) (hz : fst (U z))
      (eps : QQ_pos) ->
      Sigma (delta : QQ_pos)
            ( (w : C_c) (hw : fst (U w)) ->
              0 < abs (w - z) ->
              abs (w - z) < rat delta ->
              abs ( (f w hw - f z hz) / (w - z) - f' z hz )
                < rat eps )

-- Power series with constructive lower-bound radius.
record PowerSeries : Type where
  field
    coeff    : Nat -> C_c
    centre   : C_c
    radius   : R_c
    radius>=0 : 0 <=R_c radius
```

```
-- Sum of a power series, defined on the open disc B_R(z_0).
evaluate :
  (ps : PowerSeries) (w : C_c) ->
  abs (w - PowerSeries.centre ps) < PowerSeries.radius ps ->
  C_c

-- Theorem 4.1 (constructive identity).
identity-thm :
  (f g : PowerSeries) ->
  PowerSeries.centre f == PowerSeries.centre g ->
  PowerSeries.radius f == PowerSeries.radius g ->
  ((w : Nat -> C_c) ->
    ((k : Nat) -> abs (w k - PowerSeries.centre f)
                  < PowerSeries.radius f) ->
    ((k : Nat) -> w k != PowerSeries.centre f) ->
    ConvergesTo w (PowerSeries.centre f) ->
    ((k : Nat) -> evaluate f (w k) _ == evaluate g (w k) _) ->
    (n : Nat) ->
      PowerSeries.coeff f n == PowerSeries.coeff g n)
```

Listing 3: Proposed module signature.

A.1 API design notes

The API follows three principles:

1. *No* lim sup. The radius of convergence is a *lower bound* R, supplied as data, not derived. This avoids the constructive black-hole of lim sup of a real sequence.

2. *Open subtypes are h-prop-valued.* The opacity predicate $U : \mathbb{C}_c \to \text{hProp}$ ensures that $\Sigma_z U(z)$ is an h-set.

3. *Holomorphy carries its derivative as data.* Rather than making f' a derived notion (which constructively requires a choice principle), the record bundles f and f' and the ε–δ witness.

B A Worked Example: $\frac{1}{1-z}$ on $B_1(0)$

The geometric series $\sum_n z^n$ is the canonical example of a power series with positive radius of convergence and a known holomorphic limit. We sketch how each of the five numbered statements specialises to this example.

B.1 Setup

Let $c_n := 1$ for every $n : \mathbb{N}$, $z_0 := 0$, $R := 1 - 2^{-10}$. Then $(c, 0, R)$: PowerSeries and the sum is the geometric-series function $f(w) = \sum_{n=0}^{\infty} w^n$.

B.2 Density (Theorem 3.1)

For $z = 0.7 + 0.3i \in B_R(0)$ and $\varepsilon = 2^{-5}$, the proof of Theorem 3.1 produces dyadic $p, q : \mathbb{D}$ with $|z - (p + iq)| < \varepsilon$. Direct computation: $p = 23/32 = 0.71875$, $q = 5/16 = 0.3125$ gives $|z - (p + iq)| = |-0.01875 + (-0.0125)i| \approx 0.0225 < 2^{-5} = 0.03125$.

B.3 Identity (Theorem 5.1)

Let $c'_n := 1$ for every n be a second copy. Then $f \equiv g$ trivially. The proof of Theorem 5.1 requires no work in this case, but it is informative: pick the sequence $w_k := 2^{-k-1}$ (dyadic). Then $w_k \in B_R(0)$, $w_k \neq 0$, $\lim_k w_k = 0$. The induction on coefficients trivially yields $d_n = 0$ for every n since $f - g \equiv 0$ at every w_k.

B.4 Identity on overlap (Corollary 5.3)

Suppose we have a *second* representation of the geometric series, say (c'_n) with each c'_n computed independently and witnessed equal to 1. The corollary says that any other power-series representation that agrees with the geometric series on $B_{1-2^{-10}}(0)$ must have the same coefficients. This is a sanity check: there is exactly one power-series representative of $1/(1-z)$ at the origin.

B.5 Universal cover (Theorem 7.2)

The geometric series is holomorphic on $B_1(0) \subset \mathbb{C}^*$, but extends to no larger disk. To analytically continue past $z = 1$ (the simple pole), one must use the function $1/(1 - z)$ directly, which is holomorphic on $\mathbb{C}_c \setminus \{1\}$. The universal cover of $\mathbb{C}_c \setminus \{1\}$ (not $\mathbb{C}^*$!) is the relevant object here; it admits the same HIT presentation as Theorem 7.2 mutatis mutandis (replace exp by a translated logarithm centred at 1).

B.6 Path lifting (Theorem 6.3)

For analytic continuation of $1/(1 - z)$ along the path $\gamma(t) = (1 + t)/2$ (which crosses $z = 1$ at $t = 1$), the path-lifting lemma fails because $\gamma(1) = 1$ is the pole. This is the correct constructive obstruction: the lift does not exist because the germ is undefined at the singularity. The lemma correctly produces a partial lift on $[0, 1)$ and signals failure at $t = 1$, in keeping with constructive analysis tradition where "the lift does not exist" is positive information.

C Summary Table of Theorem Status

Theorem	Status	Dependencies
3.1 (density)	provable now	universal property of $\mathbb{R}_c$
5.1 (identity for power series)	provable now (BM)	holomorphy of power series; continuity at the centre
5.3 (identity on overlap)	provable now (BM)	Theorem 5.1
6.4 (monodromy)	formulation; conjectural	path-lifting; bundle-of-germs HIT
7.2 (universal cover of $\mathbb{C}^*$)	formulation; partial	constructive exp; inverse function theorem on disks
6.3 (path lifting)	statement only	`Cubical.HITs.CauchyReals` upstream
5.4 (local-to-global identity)	sketch; conjectural	constructive distance to boundary (located complement)

The single chokepoint is the bottom of the dependency graph: `Cubical.HITs.CauchyReals` not yet upstream. Once that merges, the BM-tagged statements are immediately formalisable, and the formulation-only statements become formalisable in stages (monodromy is the largest single piece, the universal cover is self-contained but uses exp).

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