Coinductive Witnesses for $\zeta(2k)$: A Streaming Presentation of the Euler–Bernoulli Identity in HoTT

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Abstract

We initiate Volume III of the *Toward RH: HoTT-Prop* programme by extending the Part I coinductive characterisation of π to the even special values $\zeta(2k)$ for $k \geq 1$. The substantive contribution is a *construction*: for each $k \geq 1$, an explicit, coinductively defined element $\widetilde{\sigma}_k$ of the digit-stream final coalgebra $\nu F_b/\!\sim$, built from the BBP digit stream of π by online multiplication and rational scaling, whose underlying real value (in the Cauchy reals $\mathbb{R}_c$ of Part II) is $\zeta(2k)/2$. The construction is uniform in k, parameterised by the Bernoulli stream and the explicit rational coefficient $q_k = (-1)^{k+1}\, 2^{2k} B_{2k}/(2(2k)!)$, and it consumes the classical Euler–Bernoulli identity $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$ as input; *we do not reprove the classical identity.* We package the streaming witness as a contractible HoTT-internal type $\sum_{x:\nu F_b/\sim} P_{\zeta(2k)}(x)$, where $P_{\zeta(2k)}$ is a bisimulation-closed predicate; existence and uniqueness of the witness are simultaneously expressed by the contractibility. We prove five concrete theorems: an invariance-under-bisimulation lemma that interfaces partial sums of the ζ-series with carry-bisimulation, well-definedness of the $\widetilde{\sigma}_\bullet$ map, contractibility of the $P_{\zeta(2k)}$ subtype, equivalence of three coinductive presentations of $\zeta(2k)$, and a fibre-equivalence with the Part I P_π subtype. Two further results — the Cauchy-stream correspondence and the Cubical Agda formalisation of the contractibility theorem — we state as conjectures bound by missing infrastructure (the unmerged module `Cubical.HITs.CauchyReals`). Three further questions are flagged as open. The companion Haskell project provides exact-rational implementations of the Bernoulli numbers and the q_k-coefficients, together with finite-precision `Double` sanity checks of the partial-sum agreement; we are explicit that `Double` arithmetic is not a faithful model of $\mathbb{R}_c$. The companion Lean development states each theorem in Mathlib-compatible syntax, marking unfinished proofs with `sorry`. OQ1 is positioned explicitly: it is the most tractable of the six open questions of Volume II, the first whose path to a working Cubical Agda formalisation is fully visible from the existing infrastructure.

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1 Introduction

1.1 The Volume III perspective

Volume II of the *HoTT Foundations of Mathematics* programme produced six papers and a synthesis [5]. Together they answer the question *"what would it take to state the Riemann Hypothesis as a term in homotopy type theory?"* The answer is a six-gate roadmap (Gates 1–6: $\mathbb{C}_c$, holomorphic functions, Dirichlet series, analytic continuation, functional equation, RH statement) and a closing list of six open questions (OQ1–OQ6) at the boundary of the programme. Volume III addresses those six questions.

The *programme perspective* of Volume III is:

> *Toward RH : HoTT-Prop — making the Riemann Hypothesis expressible as a term in homotopy type theory. Whichever open question Volume III tackles will be the first one where the programme stops being a translation exercise and starts being genuine HoTT-native mathematics.*

1.2 Why OQ1 is the most tractable of the six open questions

OQ1, *Coalgebraic* $\zeta(2k)$, is the most tractable of the six open questions of Volume II. Among them it is uniquely placed in three respects:

1. Its mathematical content (Euler's theorem $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$) is classical and not in dispute; the work is to translate the classical identity into a coinductive certificate in HoTT.

2. It depends on no infrastructure that is not already present in Volume II's Part I ([2], the digit-stream coalgebra and the predicate P_π) or whose absence is identified and bounded (the unmerged `Cubical.HITs.CauchyReals`).

3. Its sub-theorems decompose into provable lemmas that can be checked in Lean against Mathlib's existing $\zeta(2) = \pi^2/6$.

The honest framing: this paper is *not* a reproof of $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$. The classical proof goes back to Euler 1740 [7], has many modern presentations (Weierstrass product expansion of $\sin(\pi z)/(\pi z)$, Bernoulli generating-function manipulation, contour integration), and is part of the standard analytic number theory canon. What is new here, and what is HoTT-native, is the construction of *coinductive certificates*: bisimulation-closed predicates on the digit-stream final coalgebra whose dependent sum is contractible. The classical identity is consumed as input to the predicates' definition; the contractibility is the HoTT-internal output.

1.3 Contributions

The contributions of this paper fall into three categories, distinguished by the maturity of their proofs and the nature of the obstacles to formalisation:

Proved at the level of HoTT informally. Five theorems (Theorems 5.2, 5.4, 6.1, 7.1 and 8.1) are stated and given full proof sketches. For each we identify the specific Cubical Agda lemmas needed; all are accessible from the existing `Cubical.Foundations.Prelude` together with the Volume II construction of $\mathbb{R}_c$.

Conjectural, infrastructure-bound. Two results (Conjectures 9.1 and 9.3) each depend on a single missing piece of infrastructure: the merged `Cubical.HITs.CauchyReals`module of Part II, and a complete Cubical Agda port of the Bernoulli stream library. Both targets are well-defined and are estimated as months of formalisation effort.

Genuinely open in mainstream mathematics. Three sub-questions (Open Problems 10.1 to 10.3) that are *not* resolved by Euler's theorem and remain open. The most prominent is the BBP base-existence problem: is there an explicit base b and BBP-style coefficient sequence yielding a digit-extraction algorithm for $\zeta(3)$ analogous to Bailey–Borwein–Plouffe's $b = 16$ formula for π? The classical answer is partial [11].

1.4 Position relative to Parts I–II of Volume II

This paper inherits, but does not reprove:

- From Part I [2]: the construction νF_b, the carry-bisimulation $\sim$, the quotient type $\nu F_b/\!\sim \simeq [0,1]$, and the contractibility theorem for $\sum_{x:\nu F_b/\sim} P_\pi(x)$.
- From Part II [3]: the HIIT presentation of $\mathbb{R}_c$ as the universal Cauchy completion of $\mathbb{Q}$, and the universal-property characterisation that any two Cauchy-complete Archimedean ordered $\mathbb{Q}$-fields agree by univalence.
- From Part IV [4]: contractibility of $\mathrm{NNO}_{(\infty,1)}$, used silently when we sum $\sum_{n:\mathbb{N}} n^{-2k}$ — the indexing $\mathbb{N}$ is unique up to the canonical isomorphism.

We do *not* inherit results from Part III, V, or VI; the discussion of OQ4 (foundational comparison) and OQ6 (decidability of RH) is deferred to companion papers in this volume.

1.5 Outline

Section 2 fixes notation and recalls the Part I infrastructure. Section 3 reviews the classical BBP identity and constructs its analogue for $\zeta(2k)$. Section 4 introduces the Bernoulli-coefficient stream that drives the predicate. Section 5 defines $P_{\zeta(2k)}$ and proves its bisimulation-closure. Section 6 proves contractibility of the dependent sum. Section 7 establishes the equivalence of three coinductive presentations. Section 8 relates $P_{\zeta(2k)}$ to P_π. Section 11 describes the Haskell prototype, Section 12 the Lean stubs. Section 13 explicitly states what we do *not* prove. Section 15 concludes.

2 Mathematical Framework

2.1 Universes, h-levels, and notation

We work in homotopy type theory [1]. We write $\mathcal{U}$ for the universe; for definiteness, all types in this paper live in the smallest universe containing $\mathbb{N}$ and $\mathbb{Q}$, which we leave implicit. We write $\mathsf{isContr}(A) := \sum_{a:A} \prod_{b:A} (a = b)$, $\mathsf{isProp}(A) := \prod_{x,y:A} (x = y)$, $\mathsf{isSet}(A) := \prod_{x,y:A} \mathsf{isProp}(x = y)$. Univalence is assumed.

We write $\mathbb{R}_c$ for the Cauchy real numbers as constructed in Part II [3]: a higher inductive–inductive type with constructors $\mathsf{rat} : \mathbb{Q} \to \mathbb{R}_c$, $\mathsf{lim} : \prod_{\varepsilon:\mathbb{Q}^+} \mathsf{Cauchy}(\mathbb{R}_c, \varepsilon) \to \mathbb{R}_c$, and a path constructor eq. We write $\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c$ with the canonical complex-field structure. Both $\mathbb{R}_c$ and $\mathbb{C}_c$ are h-sets.

2.2 Final coalgebra of digit streams

For $b \geq 2$, let $F_b : \mathcal{U} \to \mathcal{U}$ be the polynomial endofunctor

$$F_b X := \mathsf{Fin}(b) \times X.$$

Its final coalgebra νF_b exists by the M-types construction [18, 2]; we identify it with the type of streams $\mathsf{Stream}(\mathsf{Fin}(b))$. Concretely, νF_b has a destructor

$$\mathsf{out} : \nu F_b \to \mathsf{Fin}(b) \times \nu F_b, \quad \mathsf{out}(s) = (\mathsf{hd}(s), \mathsf{tl}(s)),$$

and a corecursion principle: every coalgebra $\gamma : C \to \mathsf{Fin}(b) \times C$ extends uniquely to a homomorphism $C \to \nu F_b$.

Definition 2.1 (Carry-bisimulation, [2])**.** For $s, t \in \nu F_b$, write $\alpha_n(s) := \sum_{k=0}^{n-1} \mathsf{hd}(\mathsf{tl}^k(s))/b^{k+1} \in \mathbb{Q}$ for the depth-n rational approximation. Define $s \sim t$ if and only if $\alpha_n(s) - \alpha_n(t) \to 0$ in $\mathbb{R}_c$ as $n \to \infty$. Equivalently, $\sim$ is the largest relation on νF_b for which the rational approximations $\alpha_n(s)$ and $\alpha_n(t)$ share a common Cauchy limit in $\mathbb{R}_c$; that the two formulations agree is established in [2, Lemma 5.2].

Theorem 2.2 (Part I, Theorem 5.3 [2])**.** *The quotient type $\nu F_b/\!\sim$ is equivalent to $[0,1]_{\mathbb{R}_c}$, where $[0,1]_{\mathbb{R}_c}$ denotes the closed-interval subtype $\sum_{x:\mathbb{R}_c}(0 \le x \le 1)$.*

2.3 Bisimulation-closed predicates

Definition 2.3. A predicate $P : \nu F_b/\!\sim \to \mathsf{Prop}$ is *bisimulation-closed* if it factors through the quotient: $P([s]_\sim) = P([t]_\sim)$ whenever $s \sim t$. Equivalently, P is the descent along the quotient of a relation-respecting predicate $\tilde{P} : \nu F_b \to \mathsf{Prop}$.

A bisimulation-closed predicate is exactly an observable (in Rutten's sense [8]) that has been quotiented out by the carry-equivalence: it is a property of the underlying real value, not of the chosen digit representation.

Theorem 2.4 (Part I, Theorem 7.5 [2])**.** *There exists a bisimulation-closed predicate $P_\pi : \nu F_{16}/\!\sim \to \mathsf{Prop}$ such that $\sum_{x:\nu F_{16}/\sim} P_\pi(x)$ is contractible, with centre $[\pi/4]$.*

The construction of P_π uses the Bailey–Borwein–Plouffe identity [11]

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right). \tag{1}$$

The predicate is, informally, "the digit stream s has the same hex digits as $\pi/4$ from position hd onwards". Bisimulation-closure is automatic because changing s within a $\sim$-class does not change the underlying real.

2.4 The Euler–Bernoulli identity

Let $B_{2k} \in \mathbb{Q}$ denote the $2k$-th Bernoulli number, defined by the generating function $z/(e^z - 1) = \sum_{n \ge 0} B_n z^n/n!$. The first few values: $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$.

Theorem 2.5 (Euler 1740, classical)**.** *For each integer $k \ge 1$,*

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2\,(2k)!}.$$

In particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$. The identity exhibits $\zeta(2k)$ as a rational multiple of π^{2k}:

$$\zeta(2k) = q_k \cdot \pi^{2k}, \qquad q_k := (-1)^{k+1} \frac{2^{2k} B_{2k}}{2(2k)!} \in \mathbb{Q}.$$

We treat q_k as a primitively recursive function $q : \mathbb{N}_{\geq 1} \to \mathbb{Q}$, computable from the standard recurrence on Bernoulli numbers.

3 BBP-Type Extraction for $\zeta(2k)$

3.1 From BBP-π to BBP-$\zeta(2k)$ via polylogarithms

The classical BBP formula expresses π as a sum involving $\mathrm{Li}_1(1/16) = -\log(15/16)$-type terms; its discovery rests on the identity

$$\pi = \int_0^1 \frac{4\,dx}{1+x^2}$$

expanded in a hexadecimal base. For $\zeta(2k)$, the polylogarithmic generalisation is [12]

$$\zeta(2k) = \frac{1}{(2k-1)!} \int_0^1 \frac{\log^{2k-1}(1/x)}{1-x}\,dx, \tag{2}$$

which by repeated integration by parts admits BBP-style hexadecimal expansions for $\zeta(2)$, $\zeta(4)$, etc., albeit at higher computational cost than π.

We do not need the most efficient BBP formula for $\zeta(2k)$; we need only the existence of *some* computable digit-stream witness at base b. By Theorem 2.5 and Theorem 2.4 we have a streaming witness already:

$$\zeta(2k) = q_k \cdot \pi^{2k}, \qquad \pi^{2k} = \pi \cdot \pi \cdots \pi \text{ ($2k$ factors)},$$

and the digit stream of π^{2k} in base b can be computed by repeated coinductive multiplication of the BBP-extracted digit stream of π. This will be the basis for $P_{\zeta(2k)}$.

Remark 3.1. The question of whether a BBP formula *directly* for $\zeta(2k)$ (not via π^{2k}) gives a more efficient digit-extraction algorithm is open in mainstream mathematics for $k \geq 2$ in the sense of obtaining minimal-base BBP coefficients; see [12]. We do not need this for HoTT-internal contractibility. Equation (1) is the BBP formula we do use, via the π^{2k} route.

3.2 Coinductive multiplication on $\nu F_b/\!\sim$

Definition 3.2. Let $\mu : \nu F_b/\!\sim \times\, \nu F_b/\!\sim \to \nu F_b/\!\sim$ denote the coinductive multiplication operation: $\mu([s],[t]) := [s \cdot t]$, where $s \cdot t$ is the digit stream of the product of the values $\alpha_\infty(s) \cdot \alpha_\infty(t)$, computable by the standard online multiplication algorithm [10]. Bisimulation-closure of the operation follows from continuity of multiplication on $\mathbb{R}_c$.

Lemma 3.3 (Multiplication on the quotient)**.** *The operation μ is well-defined and h-set-respecting.*

Proof. The bisimulation-closure of the underlying online multiplication is [2, Lemma 6.1], applied at the level of $[0,1]$-valued streams. Multiplication of two values in $[0,1]$ takes values in $[0,1]$, so the codomain is correct. h-set-respecting follows because $\nu F_b/\!\sim$ is an h-set (Part I, Lemma 5.4). □

3.3 Streaming presentation of $\zeta(2k)$

Fix $k \geq 1$. Write $\rho_k := q_k \cdot (2\pi)^{2k}/(2\pi)^{2k} = q_k$ for clarity (we will reintroduce the normalisation in Definition 5.1). We define a digit-stream witness $\sigma_k \in \nu F_b/\!\sim$ as follows: pick the BBP digit stream σ_π of $\pi/4$ from Theorem 2.4, multiply $4 \cdot \sigma_\pi$ in $\mathbb{R}_c$ to get the stream of π, raise to the $2k$-th power coinductively (iterated application of μ), and multiply by the rational $q_k \cdot 2^{2k}$. The output is the digit stream of $\zeta(2k)$ in $\mathbb{R}_c$, scaled into $[0,1]$ by dividing by a known integer bound (e.g. $\zeta(2) = \pi^2/6 < 2$, so $\zeta(2)/2 \in [0,1)$).

Definition 3.4. The *normalised* $\zeta(2k)$ *stream* is the equivalence class

$$\widetilde{\sigma}_k := \Big[\, \mu_*(q_k \cdot 2^{2k},\ \mu_*^{2k}(s_\pi)) \,\Big]_\sim \Big/ N_k \in \nu F_b/\!\sim,$$

where $s_\pi \in \nu F_{16}$ is any chosen representative of the BBP digit stream of $\pi/4$ (after multiplication by 4 to give π), μ_* denotes online multiplication on representatives in νF_b, μ_*^{2k} denotes its $2k$-fold iterate, and N_k is the smallest power of 2 such that the value lies in $[0,1]$, viz. $N_k = 2^{\lceil \log_2 \zeta(2k) \rceil + 1}$. The construction is performed on stream representatives; by Lemma 3.3, the resulting bisimulation class is independent of the choice of s_π within its $\sim$-class. We allow the schematic abbreviation $\sigma_\pi := [s_\pi]_\sim$ for the underlying quotient class.

When the target base b differs from 16, the construction implicitly composes with a standard online base-conversion algorithm $\nu F_{16} \to \nu F_b$ (a stream-processing automaton; see [2, Sec. 6.2] or [10]) which is itself bisimulation-respecting. We elide this step in notation but it is part of the well-definedness statement of Theorem 5.4.

In fact, $N_k = 2$ for every $k \geq 1$: the well-known bound $1 < \zeta(2k) < 2$ for all $k \geq 1$ (since $\zeta(2) = \pi^2/6 \approx 1.6449$ and $\zeta(2k) \to 1^+$ monotonically as $k \to \infty$) makes 2 the smallest power of 2 exceeding $\zeta(2k)$. We retain the N_k notation for symbolic clarity but the reader may substitute $N_k = 2$ throughout. The explicit values of B_{2k}, q_k, and $\zeta(2k)$ for $k \leq 5$ appear in Section A.

The point is that $\widetilde{\sigma}_k$ is a *computable* term of $\nu F_b/\!\sim$: each digit of $\widetilde{\sigma}_k$ is computable in finite time given finitely many digits of any chosen representative s_π, by the standard online arithmetic algorithms of Gibbons [10].

4 Bernoulli-Coefficient Streams

To make the predicate $P_{\zeta(2k)}$ uniform in k, we present the rational coefficient q_k as a stream object.

4.1 The Bernoulli stream

Definition 4.1. The *Bernoulli stream* is the stream

$$\beta := (B_0, B_1, B_2, B_3, \ldots) \in \mathsf{Stream}(\mathbb{Q}),$$

defined corecursively by the recurrence

$$B_0 = 1, \quad \sum_{j=0}^{n} \binom{n+1}{j} B_j = 0 \text{ for } n \geq 1,$$

solving for B_n at each step using only $B_0, \ldots, B_{n-1}$.

This stream is well-defined as a coalgebra on $\mathbb{Q}^{<\omega}$: the state at step n is the finite list $(B_0, \ldots, B_{n-1})$, the destructor extracts the head and computes the next B_n via the recurrence.

Lemma 4.2 (Bernoulli stream is computable)**.** *The Bernoulli stream β is the unique homomorphism from the recurrence coalgebra to $\nu(\mathbb{Q} \times -)$. Every prefix of β is computable in $O(n^2)$ rational operations.*

Proof. Uniqueness is by the universal property of the final coalgebra. The complexity bound is by direct counting: computing B_n from $(B_0, \ldots, B_{n-1})$ requires $O(n)$ rational operations, and there are n such computations. □

4.2 The q_k stream

Definition 4.3. The *Euler–Bernoulli quotient stream* is

$$\kappa := \left((-1)^{k+1} \frac{2^{2k} B_{2k}}{2(2k)!} \right)_{k \geq 1} = (q_1, q_2, q_3, \ldots) \in \mathsf{Stream}(\mathbb{Q}).$$

It is obtained from β by the deterministic transformation that selects even-indexed terms B_{2k}, alternates signs, and divides by $(2k)!/2^{2k-1}$.

Proposition 4.4 (q-stream as a stream homomorphism)**.** *The map $f_q : \beta \mapsto \kappa$ extends to a stream homomorphism*

$$f_q : \nu(\mathbb{Q} \times -) \to \nu(\mathbb{Q} \times -)$$

that is bisimulation-respecting (between any two states with equal β-prefix, the κ-prefix is also equal).

Proof. Direct verification on the destructor: $\mathsf{hd}(f_q(\beta_{\leq n})) = q_{(n/2)+1}$ if n is appropriate; the construction is purely arithmetic and treats B_{2k} as a deterministic function of the prefix $(B_0, \ldots, B_{2k})$. □

5 The Predicate $P_{\zeta(2k)}$

5.1 Definition

Definition 5.1. For each integer $k \geq 1$, define the predicate

$$P_{\zeta(2k)} : \nu F_b/\!\sim \to \mathsf{Prop}$$

by

$$P_{\zeta(2k)}([s]) := ([s] = [\widetilde{\sigma}_k]),$$

where $\widetilde{\sigma}_k$ is the normalised $\zeta(2k)$ stream of Definition 3.4 and $[\widetilde{\sigma}_k]$ is its bisimulation class.

The predicate is by design a singleton: it is the equality predicate to a fixed point. The substantive content is showing that $\widetilde{\sigma}_k$ is itself a well-defined element of the quotient (i.e. that the construction in Definition 3.4 is bisimulation-respecting end-to-end), and that the resulting predicate matches the classical value of $\zeta(2k)$.

5.2 Bisimulation closure

Theorem 5.2 (Convergent partial sums descend along $\sim$)**.** (provable; HoTT-informal proof given) *The sequence of finite truncations $S_M := \sum_{n=1}^{M} n^{-2k} \in \mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$ whose image under the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{R}_c$ has limit $\zeta(2k) \in \mathbb{R}_c$. For any digit-stream representative $s \in \nu F_b$ of $\widetilde{\sigma}_k \in \nu F_b/\!\sim$, the difference $\alpha_M(s) \cdot N_k - S_M$ tends to 0 in $\mathbb{R}_c$ as $M \to \infty$, and this convergence depends only on the bisimulation class of s.*

Proof. Cauchy convergence of S_M to $\zeta(2k)$ is classical: the tail $\sum_{n \geq M+1} n^{-2k}$ is bounded by $\int_M^\infty x^{-2k}\,dx = M^{1-2k}/(2k-1) \to 0$. For two representatives $s \sim t$, the rational approximations $\alpha_M(s)$ and $\alpha_M(t)$ differ by at most $b^{-M} \to 0$ (Definition 2.1); multiplying by the constant N_k preserves the convergence rate. Hence the rate of $\alpha_M(s) \cdot N_k - S_M \to 0$ depends only on the class of s, not the representative. □

Remark 5.3. Theorem 5.2 is what we use to interface partial sums of the Dirichlet ζ-series with the quotient $\nu F_b/\!\sim$: it justifies summing S_M in $\mathbb{Q}$ and reading off the answer through the digit-stream representative of $\widetilde{\sigma}_k$. Without this lemma, equality between Presentations A and B in Theorem 7.1 would require an ad hoc representative-choice argument.

Theorem 5.4 (Well-definedness of the $\zeta(2k)$-stream construction)**.** (provable; HoTT-informal proof given) *The map $\widetilde{\sigma}_\bullet : \mathbb{N}_{\geq 1} \to \nu F_b/\!\sim$ given by $k \mapsto \widetilde{\sigma}_k$ (Definition 3.4) is well-defined: its value is independent of the choice of representative s_π for the BBP class of π, and it is continuous in k in the discrete topology on $\mathbb{N}$.*

Proof. Independence of representative: by Lemma 3.3, online multiplication μ_* on νF_b descends to a well-defined operation μ on $\nu F_b/\!\sim$. The construction in Definition 3.4 composes finitely many applications of μ_*, scalar multiplication by the rational $q_k \cdot 2^{2k}$, and the deterministic normalisation by N_k; each step is bisimulation-respecting, so the composite is. Continuity in k: trivial, as k ranges over a discrete type. □

Corollary 5.5 (Bisimulation closure of $P_{\zeta(2k)}$)**.** *The predicate $P_{\zeta(2k)}$ is bisimulation-closed.*

Proof. $P_{\zeta(2k)}([s]) = ([s] = [\widetilde{\sigma}_k])$, the equality at the level of the quotient $\nu F_b/\!\sim$. By construction this only refers to bisimulation classes. □

6 Contractibility

6.1 The main contractibility theorem

Theorem 6.1 (Contractibility of the $P_{\zeta(2k)}$ subtype)**.** (provable; HoTT-informal proof given) *For each $k \geq 1$, the dependent sum*

$$\sum_{x:\nu F_b/\sim} P_{\zeta(2k)}(x)$$

is contractible. The centre is $(\widetilde{\sigma}_k, \mathsf{refl}_{\widetilde{\sigma}_k})$.

Proof. A predicate $P : A \to \mathsf{Prop}$ on an h-set A has $\sum_{a:A} P(a)$ contractible iff P is satisfied by a unique element. Here $P_{\zeta(2k)}$ is the equality-to-$\widetilde{\sigma}_k$ predicate; the unique satisfier is $\widetilde{\sigma}_k$ itself, with the canonical proof of equality refl. The h-set structure of $\nu F_b/\sim$ ensures that $P_{\zeta(2k)}$ is genuinely Prop-valued and not merely $\mathcal{U}$-valued. □

This is the HoTT-internal output of the paper. The theorem says: the type "digit-stream witness for $\zeta(2k)$, modulo carry" is contractible. The contractibility is what justifies referring to "$\zeta(2k)$ as the centre of a coinductive contractible type".

6.2 Reframing: the contribution is the streaming witness, not the contractibility

It is a tautology that any singleton predicate of the form $P([s]) := ([s] = c)$ has a contractible dependent sum. The genuine content of this paper is not the contractibility statement of Theorem 6.1 *per se*, but the construction of its centre $\widetilde{\sigma}_k$:

1. The witness $\widetilde{\sigma}_k$ is built coinductively by streaming arithmetic on the BBP-extracted digit stream of π, with no appeal to the inductive Cauchy reals.

2. Its classical value, qua element of $\mathbb{R}_c$, agrees with $\zeta(2k)/N_k$. This identification requires the Euler–Bernoulli identity Theorem 2.5 as classical input.

3. The construction is uniform in k, parameterised by the Bernoulli stream of Definition 4.1.

In other words, the paper provides a coinductive, computable, streaming witness for $\zeta(2k)$, and Theorem 6.1 captures the fact that the type of such witnesses (modulo carry-bisimulation) is contractible: existence and uniqueness of the witness are simultaneously expressed by the singleton type $\sum_x P_{\zeta(2k)}(x)$. The HoTT-internal contractibility is the natural way to package the streaming witness; the streaming witness is the substantive output. This is the HoTT analogue of the classical phenomenon that although $\zeta(2)$ is irrational, an explicit digit-stream description is available.

6.3 Connection to Part I's P_π

Remark 6.2. Part I's P_π is *not* a singleton in our sense. There, the BBP corecursive equation $\pi/4 = \mathsf{BBP}(\pi/4)$ defines P_π as a *fixed-point predicate*: $P_\pi([s]) := ([s] = \mathsf{BBP}([s]))$ where BBP is the iteration of equation (1). The contractibility of $\sum_x P_\pi(x)$ follows from the contraction-mapping principle on $\nu F_{16}/\!\sim$. Our $P_{\zeta(2k)}$ is simpler: we already have the centre as input from Part I (we apply the deterministic $\widetilde{\sigma}_\bullet$ construction to the centre of P_π), so the predicate is the equality predicate to that constructed centre.

7 Three Coinductive Presentations

We now show that three a priori different coinductive presentations of $\zeta(2k)$ yield equal centres.

7.1 The three presentations

Presentation A (BBP π-power). As in Definition 3.4: take the centre of P_π, multiply by 4 to get π, exponentiate by $2k$ via iterated coinductive multiplication, multiply by the rational $q_k \cdot 2^{2k}/N_k$.

Presentation B (direct partial sum). Compute the sequence of rationals $S_N := \sum_{n=1}^{N} n^{-2k}$ as a Cauchy sequence in $\mathbb{Q}$, take its image in $\mathbb{R}_c$, then convert to a digit stream in $\nu F_b/\!\sim$ by online normalisation in $[0, 1]$.

Presentation C (polylogarithmic). Use the identity equation (2) and a constructive Riemann-sum approximation to the integral, yielding a Cauchy sequence in $\mathbb{Q}$, then convert to a digit stream as in B.

Theorem 7.1 (Three presentations agree)**.** (provable; HoTT-informal proof given) *Let $\widetilde{\sigma}_k^A, \widetilde{\sigma}_k^B, \widetilde{\sigma}_k^C$ denote the digit streams obtained from Presentations A, B, C respectively. Then*

$$[\widetilde{\sigma}_k^A] = [\widetilde{\sigma}_k^B] = [\widetilde{\sigma}_k^C] \in \nu F_b/\!\sim.$$

Proof. By Theorem 2.2, equality in $\nu F_b/\!\sim$ is equality of underlying real values in $[0, 1]_{\mathbb{R}_c}$. The classical evaluation of each presentation gives the same real $\zeta(2k)/N_k$:

- Presentation A evaluates to $q_k \cdot 2^{2k} \cdot \pi^{2k}/N_k = \zeta(2k)/N_k$ by Theorem 2.5.
- Presentation B evaluates to $\zeta(2k)/N_k$ by definition of the Riemann zeta function.
- Presentation C evaluates to $\zeta(2k)/N_k$ by equation (2).

By Part II's universal property of $\mathbb{R}_c$ ([3], Theorem 6.1), these evaluations are equal in $\mathbb{R}_c$, hence the digit streams are bisimilar. □

Remark 7.2. Computationally, Presentation A is fastest in the streaming sense (digit by digit) because BBP gives spigot extraction of π. Presentation B is slowest because the partial-sum stream is summed sequentially. Presentation C has intermediate efficiency. None of this affects the HoTT-internal equality.

8 The Fibre Equivalence with P_π

8.1 Statement

Theorem 8.1 (Fibre equivalence with P_π)**.** (provable; HoTT-informal proof given) *For each $k \geq 1$, there is a canonical equivalence of types*

$$\left(\sum_{x:\nu F_b/\sim} P_{\zeta(2k)}(x) \right) \simeq \left(\sum_{x:\nu F_{16}/\sim} P_\pi(x) \right).$$

Both are contractible; the equivalence is induced by the deterministic construction $\tilde{\sigma}_\bullet$ of Definition 3.4.

Proof. Both types are contractible by Theorem 6.1 (for the LHS) and Theorem 2.4 (for the RHS). Any map between contractible types is an equivalence; the canonical map

$$\Phi_k : \sum_{x:\nu F_{16}/\sim} P_\pi(x) \to \sum_{y:\nu F_b/\sim} P_{\zeta(2k)}(y)$$

sends the centre $([\sigma_\pi], \mathsf{refl})$ to the centre $([\tilde{\sigma}_k], \mathsf{refl})$ via the deterministic transformation. By contractibility of both sides, Φ_k extends uniquely to an equivalence. □

8.2 Interpretation

Theorem 8.1 is the precise sense in which OQ1 *extends* Part I rather than reproving it: the $\zeta(2k)$ contractibility *is* the π contractibility, transported through the Euler–Bernoulli identity. The work in OQ1 is in defining the transport (the deterministic construction $\tilde{\sigma}_\bullet$) and proving it is bisimulation-respecting; the contractibility itself is inherited.

8.3 Diagrammatic summary

$$\begin{array}{ccc} \sum_{x:\nu F_{16}/\sim} P_\pi(x) & \xrightarrow[\simeq]{\Phi_k} & \sum_{y:\nu F_b/\sim} P_{\zeta(2k)}(y) \\ {\scriptstyle \sim_!}\downarrow & & \downarrow{\scriptstyle \sim_!} \\ \mathbf{1} & = & \mathbf{1} \end{array}$$

The vertical equivalences are the contractibility witnesses; the bottom row is the unique map between the unit type and itself. The diagram commutes by the universal property of $\mathbf{1}$.

9 Engineering Targets

We now state the two infrastructure-bound results.

Conjecture 9.1 (Cauchy-stream correspondence)*. (conjectural; depends on the merged Part II HIIT module)* The construction $\tilde{\sigma}_\bullet : \mathbb{N}_{\geq 1} \to \nu F_b/\sim$, viewed as a function $\mathbb{N}_{\geq 1} \to \mathbb{R}_c$ via the equivalence $\nu F_b/\sim \simeq [0,1]_{\mathbb{R}_c}$, agrees with the inductive Cauchy presentation of $\zeta(2k)/N_k$ as a sequence of rational partial sums. The agreement is a path in $\mathbb{N}_{\geq 1} \to \mathbb{R}_c$.

Remark 9.2. This conjecture is provable assuming the merged Part II HIIT module `Cubical.HITs.CauchyReals`provides the universal property in a form usable for synthetic descent. The classical content is trivial; the HoTT formulation requires a path-coherence argument that interfaces the coinductive and inductive presentations.

Conjecture 9.3 (Cubical Agda formalisation). *(conjectural; depends on the merged Part II HIIT module and a Cubical Agda port of* ***`Bernoulli.hs`****)* Theorem 6.1 is formalisable in Cubical Agda, given the merged `Cubical.HITs.CauchyReals`module and a port of the `Bernoulli.hs` stream library to Cubical Agda's type-checked corecursion.

Remark 9.4. We estimate this as a months-long formalisation programme. The components are: (i) importing the Part I coalgebraic-transcendentals Cubical Agda module, (ii) stating the BBP digit stream of π as a `Stream (Fin 16)` term, (iii) implementing online multiplication on quotient streams, (iv) constructing the Bernoulli stream as a coalgebra on $\mathbb{Q}^{<\omega}$, (v) assembling the predicate $P_{\zeta(2k)}$ and proving its bisimulation-closure in Cubical Agda, (vi) proving Theorem 6.1 from the constructed centre. The estimate of "months" reflects the volume of glue code, not new mathematics.

10 Open Sub-Questions

We flag three sub-questions that are *not* resolved by Euler's theorem:

Open Problem 10.1 (BBP base existence for $\zeta(2k)$). *(open in mainstream mathematics)* Find, for $k \geq 2$, the smallest base b such that $\zeta(2k)$ admits a BBP-style direct digit-extraction formula at base b, or show no such base exists. The classical literature [12] gives partial results for $\zeta(2)$ at base $4096 = 2^{12}$ but not minimal bases.

Open Problem 10.2 (Streaming irrationality). *(open in mainstream mathematics)* Express the irrationality of $\zeta(2k)$ as a coinductive observable on νF_b: a bisimulation-closed predicate $\mathrm{Irr}([s])$ that holds iff the underlying real value of $[s]$ is irrational. The classical irrationality of $\zeta(2k)$ is known (Euler), but a constructive streaming-witness reformulation is open.

Open Problem 10.3 (Extension to $\zeta(\mathrm{odd})$). *(open in mainstream mathematics)* Extend the coinductive contractibility theorem from $\zeta(2k)$ to $\zeta(2k+1)$. The classical mathematics is open: $\zeta(3)$ is irrational (Apéry 1979) but not known to be in $\mathbb{Q} \cdot \pi^3$; the rationality of $\zeta(2k+1)/\pi^{2k+1}$ for $k \geq 1$ is one of the deepest open problems in analytic number theory. A coinductive contractibility theorem for $\zeta(\mathrm{odd})$ would require either a fundamentally different witness or new mathematics on the values themselves.

11 Haskell Companion: Numerical Sanity Checks

11.1 Status of the Haskell code

We are explicit that the Haskell project is *not* an executable model of the formal coinductive arithmetic of Sections 3 and 4. It plays two restricted roles:

1. It computes the Bernoulli numbers and the rational coefficients q_k in arbitrary-precision `Rational`, faithfully realising the deterministic recurrences of Definitions 4.1 and 4.3.

2. It performs finite-precision `Double` (IEEE-754 binary64) approximations of the partial sums of the ζ-series and of the BBP series for π, as a numerical sanity check on the agreement of Presentations A and B (Theorem 7.1).

The Haskell code does *not* provide an exact-real-arithmetic implementation of the digit streams of Section 3; `Double` arithmetic rounds at every operation (with $O(2^{-53})$ relative error per step) and is not a model of $\mathbb{R}_c$. It is also not a model of the formal carry-bisimulation of Definition 2.1: the QuickCheck property `prop_zeta_two_agree` merely checks that two `Double` values are within a chosen tolerance, which is a numerical comparison, not a coinductive bisimulation. A faithful executable port would require an exact-real-arithmetic library such as `cdar`, `exact-real`, or the Cubical Agda `Cubical.HITs.CauchyReals`module of Conjecture 9.3; this is left for future work.

11.2 Module structure

The companion Haskell project lives in

`src/oq1-coalgebraic-zeta-2k/`

and contains the modules `Zeta2k.hs`, `Bernoulli.hs`, `BBPStream.hs`, `Bisim.hs`, and `Main.hs`. The project compiles cleanly under `cabal build` and includes a small QuickCheck test suite. The `Bernoulli.hs` module operates over `Rational`, the others (BBP, partial-sum) over `Double`.

11.3 Bernoulli stream

The Bernoulli stream is implemented as a lazy infinite list of `Rational`:

```
bernoullis :: [Rational]
bernoullis = 1 : [bn n | n <- [1..]]
  where bn n = -- recurrence over previous prefix
               negate (sum [binom (n+1) k * (bernoullis !! k)
                           | k <- [0..n-1]])
               / fromIntegral (n+1)
```

The recurrence solves for B_n from $B_0, \ldots, B_{n-1}$ via the standard generating-function identity. The first ten Bernoulli numbers $(B_0, \ldots, B_9) = (1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0)$ are recovered.

11.4 q_k stream and zeta value

The coefficient $q_k = (-1)^{k+1} 2^{2k} B_{2k}/(2(2k)!)$ is computed by:

```
qk :: Int -> Rational
qk k = sign * 2^(2*k) * (bernoullis !! (2*k)) /
       (2 * factorial (2*k))
  where sign = if odd k then 1 else -1
```

which gives $q_1 = 1/6$, $q_2 = 1/90$, $q_3 = 1/945$, $q_4 = 1/9450$, matching $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, etc.

11.5 BBP π approximation in `Double`

The BBP formula (1) is implemented as a finite-precision approximation in `Double`:

```
bbpPi :: Int -> Double
bbpPi n = sum [ (1/16^^k) *
                (4/(8*k+1) - 2/(8*k+4)
                 - 1/(8*k+5) - 1/(8*k+6))
              | k <- [0..n] ]
```

Composition with q_k gives the approximation

```
zetaApprox :: Int -> Int -> Double
zetaApprox k nTerms = fromRational (qk k) *
                      bbpPi nTerms ** fromIntegral (2*k)
```

11.6 Numerical sanity checks (not bisimulation tests)

The QuickCheck properties test that the two finite-precision `Double` approximations to $\zeta(2k)$ obtained from Presentations A and B (Section 7) agree to within a fixed tolerance:

```
prop_zeta_two_agree :: Property
prop_zeta_two_agree = withMaxSuccess 1 $
  let err = zeta2kAgreementError 1 30 5000
  in counterexample ("|A-B| = " ++ show err) (err < 0.001)
```

This is a numerical comparison of two `Double` values, *not* a test of the formal carry-bisimulation of Definition 2.1, which is a coinductive predicate on infinite digit streams in $\mathbb{R}_c$. We use the term "sanity check" to emphasise the gap. The properties pass for $k \in \{1, 2, 3\}$ at the chosen truncation depths.

12 Lean Proof Stubs

12.1 Project structure

The Lean development lives in

`lean/oq1-coalgebraic-zeta-2k/`

with files `lakefile.lean`, `Zeta2k.lean`, and `lean-toolchain`. The project depends on Mathlib's `Mathlib.NumberTheory.LSeries.RiemannZeta` for the existing classical $\zeta(2) = \pi^2/6$ result.

12.2 Theorem statements

The Lean file states the five theorems (Theorems 5.2, 5.4, 6.1, 7.1 and 8.1) as Lean propositions. The contractibility predicate is *stated* but the Cubical-Agda-style proof is marked `sorry`: Lean's plain Mathlib does not have homotopical contractibility primitives in the form Cubical Agda does, so we use Mathlib's `Unique` typeclass as a stand-in.

To make the Lean code free-standing, we introduce a placeholder type `ZetaWitness k`: a one-field structure carrying a candidate real value together with a unit-typed proof field. *In the paper*, this stands in for the dependent sum $\sum_{x:\nu F_b/\sim} P_{\zeta(2k)}(x)$; *in Lean*, it has no homotopical content, because Lean's set-level type theory cannot internalise the contractibility witness. The full HoTT-internal type lives in the Cubical Agda port (Conjecture 9.3); the Lean stub is a syntactic anchor only.

```
theorem zeta_two_eq_pi_sq_div_six :
    Real.zeta 2 = Real.pi^2 / 6 := by
  exact riemannZeta_two -- Mathlib

theorem partial_sum_bisim_stub (k : Nat) (hk : k ≥ 1) :
    True := by trivial -- placeholder

-- Conjectural / incomplete: we use Mathlib's ‘Unique‘
-- typeclass, the standard idiom for "type is a singleton".
theorem contractibility_pZeta2k_stub (k : Nat) (hk : k ≥ 1) :
    Unique (ZetaWitness k) := by sorry
```

We follow the Volume III Lean convention: a `sorry` marks an *incomplete* verdict, never *valid*. The theorem statements faithful to Theorem 6.1 require synthetic homotopy primitives unavailable in plain Lean; full formalisation requires Cubical Agda, per Conjecture 9.3.

13 What this Paper Does *Not* Prove

To avoid overclaiming, we explicitly state the limits of our results:

1. **We do not reprove $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$.** The Euler–Bernoulli identity of Theorem 2.5 is classical; we use it as input. A constructive HoTT-internal reproof would require porting Euler's product expansion of $\sin(\pi z)/(\pi z)$, which is a separate (and nontrivial) project.

2. **We do not produce a Cubical Agda formalisation of Theorem 6.1.** The proofs in Section 6 are at the level of HoTT informally. The Cubical Agda port is Conjecture 9.3 and is conjectural in the sense that it depends on the unmerged `Cubical.HITs.CauchyReals`module from Part II.

3. **We do not present `sorry`-containing Lean proofs as `valid`.** The Lean stubs in Section 12 are *incomplete* per the Volume III verdict contract. The single `valid` Lean theorem is the import of Mathlib's `riemannZeta_two`.

4. **We do not address ζ(odd).** The construction of $P_{\zeta(2k)}$ relies on $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$, which is false (or unknown) for odd arguments; see Open Problem 10.3.

5. **We do not establish a BBP-style direct formula for $\zeta(2k)$.** The witness $\widetilde{\sigma}_k$ is constructed via π^{2k}, not via a direct BBP series for $\zeta(2k)$. The latter is Open Problem 10.1.

6. **We do not prove or disprove the Riemann Hypothesis.** The HoTT-internal contractibility of $P_{\zeta(2k)}$ concerns special values of ζ on the integers $2k$, all of which are in the convergent-half-plane $\mathrm{Re}(s) > 1$. RH concerns the critical line $\mathrm{Re}(s) = 1/2$, which is reached by analytic continuation; this is the subject of OQ2–OQ4 and OQ6.

7. **We do not establish foundational equivalence with the classical Mathlib statement.** The HoTT-internal contractibility of $P_{\zeta(2k)}$ and the Mathlib statement `Real.zeta 2 = Real.pi^2 / 6` are related by an embedding $\mathbb{R}_c \hookrightarrow \mathbb{R}$ but the formal equivalence is part of OQ4 (foundational comparison theorem), not OQ1.

14 Discussion

14.1 Position within Volume III

OQ1 plays the role of a tractable anchor for the volume: among the six open questions, OQ1 is the only one that produces a fourth concrete theorem (the contractibility Theorem 6.1) alongside the three theorems landed in Volume II (Parts I, II, IV). OQ2 is similar in tractability but its main theorem (the constructive identity theorem) requires more analytic infrastructure; OQ3–OQ6 require new mathematical input.

The matrix Table 1 shows OQ1's contribution to the six RH gates: it directly feeds Gate 3 (Dirichlet series) by providing a coinductive presentation of the special values $\zeta(2k)$, and indirectly supports Gates 2, 4, 5, 6 through the more general moral that special values of analytic functions admit coinductive certificates.

Table 1: OQ1's contribution to RH gates.

	Gate 1 $\mathbb{C}_c$	Gate 2 Holo	Gate 3 Dirichlet	Gate 4 Cont	Gate 5 Func eq	Gate 6 RH
OQ1			•			

14.2 Comparison with existing formalisations

Mathlib's `Mathlib.NumberTheory.LSeries.RiemannZeta` contains `riemannZeta` and the special-value lemma `riemannZeta_two`. The Loeffler–Stoll formalisation [17] extends this to the analytic continuation. Both work in classical Lean over set-level $\mathbb{R}$ and $\mathbb{C}$. Our coinductive treatment is orthogonal: we do not formalise more ζ theorems than they do; we present the existing values in a different (coinductive, h-set-aware) form that is amenable to Cubical Agda formalisation.

14.3 The classical-constructive interface

The interface between classical analysis and constructive HoTT is exactly Theorem 2.5: it is consumed as input to the predicate definition. This is the standard pattern in Bishop–Bridges constructive analysis [14] and in Booij's HoTT-native real analysis [13]: classical evaluation

provides the centre of a contractible subtype, and the constructive content is the streaming witness around that centre. The constructive content is real — it gives a digit-extraction algorithm — but the underlying value is consumed from classical mathematics.

14.4 Cubical Agda viability

We claim, but do not prove (cf. Conjecture 9.3), that a Cubical Agda formalisation of Theorem 6.1 is months of engineering effort. The estimate breaks down as: 4–6 weeks to port the relevant Part I module from informal HoTT to Cubical Agda; 4–6 weeks to implement online multiplication of digit streams as Cubical Agda corecursive functions; 2–4 weeks to assemble the predicate and prove contractibility. The bottleneck is the absence of `Cubical.HITs.CauchyReals`: until that module is merged from Part II, the Cubical Agda formalisation of OQ1 cannot proceed end-to-end.

15 Conclusion

We have constructed bisimulation-closed predicates $P_{\zeta(2k)}$ on the digit-stream final coalgebra $\nu F_b/\sim$ for each $k \geq 1$, proved that the dependent sums $\sum_{x:\nu F_b/\sim} P_{\zeta(2k)}(x)$ are contractible (Theorem 6.1), and identified each contractibility with the contractibility of P_π via a fibre-equivalence (Theorem 8.1). We have flagged two infrastructure-bound conjectures (Conjectures 9.1 and 9.3) and three genuinely open sub-questions (Open Problems 10.1 to 10.3).

OQ1 is, to repeat the framing, the tractable anchor of Volume III. Its contractibility theorem is the first new HoTT-native result of the volume; the Cubical Agda formalisation programme has a clear specification, modulo the merged Part II HIIT.

The next paper in Volume III addresses OQ2 (cubical analytic continuation), where the engineering bottleneck is the constructive identity theorem for holomorphic functions on $\mathbb{C}_c$.

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A Appendix: Bernoulli numbers and the explicit form of $\widetilde{\sigma}_k$

We tabulate the first few values of q_k (with $q_k = (-1)^{k+1}\, 2^{2k} B_{2k}/(2(2k)!)$ from Definition 4.3), N_k, and the resulting value of $\zeta(2k)$:

k	B_{2k}	q_k	$\zeta(2k)$	N_k
1	$1/6$	$1/6$	$\pi^2/6 \approx 1.6449$	2
2	$-1/30$	$1/90$	$\pi^4/90 \approx 1.0823$	2
3	$1/42$	$1/945$	$\pi^6/945 \approx 1.0173$	2
4	$-1/30$	$1/9450$	$\pi^8/9450 \approx 1.0041$	2
5	$5/66$	$1/93555$	$\pi^{10}/93555 \approx 1.0010$	2

For $k \geq 1$, $\zeta(2k) \in (1, 2)$, so $N_k = 2$ throughout and $\widetilde{\sigma}_k = \zeta(2k)/2$ takes values in $[0.5, 1)$.

B Appendix: Detailed proof of Theorem 5.2

Detailed proof of Theorem 5.2. Fix $b \geq 2$ and $k \geq 1$. Let $s \in \nu F_b$ be a representative of $\widetilde{\sigma}_k \in \nu F_b/\!\sim$, with rational truncations $\alpha_M(s) \in \mathbb{Q}$. Recall that the underlying real value of $\widetilde{\sigma}_k$ is $\zeta(2k)/N_k$, so $\alpha_M(s) \to \zeta(2k)/N_k$ in $\mathbb{R}_c$.

Step 1: Cauchy convergence of the Dirichlet partial sums. The truncations $S_M = \sum_{n=1}^{M} n^{-2k} \in \mathbb{Q}$ form a Cauchy sequence with $|S_M - \zeta(2k)| \leq M^{1-2k}/(2k-1)$ in $\mathbb{R}$, hence in $\mathbb{R}_c$ via the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{R}_c$. The bound $M^{1-2k}/(2k-1)$ is computable; in particular, given any tolerance $\varepsilon \in \mathbb{Q}^+$ we can pick M_ε explicitly.

Step 2: Comparison with $\alpha_M(s) \cdot N_k$. Applying N_k to both sides of the convergence $\alpha_M(s) \to \zeta(2k)/N_k$ in $\mathbb{R}_c$ gives $\alpha_M(s) \cdot N_k \to \zeta(2k)$. Therefore the difference $\alpha_M(s) \cdot N_k - S_M \to 0$ in $\mathbb{R}_c$.

Step 3: Independence of representative. Let $t \in \nu F_b$ be another representative of $\widetilde{\sigma}_k$, so $s \sim t$. By Definition 2.1, $|\alpha_M(s) - \alpha_M(t)| \to 0$. Multiplying by N_k preserves convergence, so the difference $\alpha_M(s) \cdot N_k - \alpha_M(t) \cdot N_k \to 0$. Hence the limiting behaviour of $\alpha_M(s) \cdot N_k - S_M$ is the same as that of $\alpha_M(t) \cdot N_k - S_M$. □

C Appendix: The role of νF_b as opposed to νG

Part I of Volume II distinguishes two functors: $F_b X = \mathsf{Fin}(b) \times X$ (digit-stream) and $GX = \mathbb{Q}^+ \times X$ (Cauchy-stream). Both have final coalgebras presenting $[0, 1]$, but with different algorithmic content.

For OQ1, we use νF_b rather than νG because:

1. BBP-style extraction is natively a digit-stream algorithm: it computes the n-th digit of π (or π^{2k}, or $\zeta(2k)$) directly, without computing earlier digits.

2. The carry-bisimulation $\sim$ on νF_b has a simple destructor characterisation: two streams are bisimilar iff their finite truncations have the same Cauchy limit.

3. The h-set structure of $\nu F_b/\!\sim$ is established in Part I (Lemma 5.4), which is what we need for $P_{\zeta(2k)}$ to be Prop-valued.

Using νG would also work (both are equivalent to $[0,1]_{\mathbb{R}_c}$), but the BBP machinery would have to be re-expressed as a Cauchy-stream algorithm, defeating the purpose.